Math 214 Spring 2017
Linear Algebra HW 9 Solutions
Due Friday April 7

For all these problems, justify your answers.

1. Let $L(x, y, z) = (x - y, 3x + z, y - 2z)$. Find a formula for L^{-1}. (Do *not* leave your answer as a matrix).

   Solution: We have

$$\begin{aligned} L(1,0,0) &= (1,3,0) \\ L(0,1,0) &= (-1,0,1) \\ L(0,0,1) &= (0,1,-2) \\ A &= \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \end{aligned}$$

   and we compute

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 7 & -3 & 1 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3/7 & 1/7 & -3/7 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/7 & 2/7 & 1/7 \\ 0 & 1 & 0 & -6/7 & 2/7 & 1/7 \\ 0 & 0 & 1 & -3/7 & 1/7 & -3/7 \end{array}\right]$$

   so

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 & 1 \\ -6 & 2 & 1 \\ -3 & 1 & -3 \end{bmatrix}.$$

   Thus

$$L^{-1}(a, b, c) = \frac{1}{7}(a + 2b + c, -6a + 2b + c, -3a + b - 3c).$$

2. Let $T : \mathbb{R}^3 \to \mathcal{P}_2(x)$ be given by $T(a, b, c) = (a - c) + (b - c)x + (a + b + c)x^2$. Find T^{-1}. (Do *not* leave your answer as a matrix).

Solution: We use the standard bases of $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{1,x,x^2\}$. We have

$$\begin{aligned} T(1,0,0) &= 1+x^2 \to (1,0,1) \\ T(0,1,0) &= x+x^2 \to (0,1,1) \\ T(0,0,1) &= -1-x+x^2 \to (-1,-1,1) \\ A &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

We compute

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & 2/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right]$$

so

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Thus

$$T^{-1}(a+bx+cx^2) = \frac{1}{3}\begin{bmatrix} 2a-b+c \\ -a+2b+c \\ -a-b+c \end{bmatrix}.$$

3. ($\star$) If $f(x) \in \mathcal{P}_2(x)$ such that $f(1)=4, f(3)=7, f(4)=1$, find $f(x)$. (Hint: define an evaluation map from $\mathcal{P}_2(x)$ to $\mathbb{R}^3$).

Solution: Define a function $E : \mathcal{P}_2(x) \to \mathbb{R}^3$ by $E(f) = (f(1), f(3), f(4))$. This is linear, and we can compute the matrix:

$$\begin{aligned} E(1) &= (1,1,1) \\ E(x) &= (1,3,4) \\ E(x^2) &= (1,9,16) \\ A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}. \end{aligned}$$

We can invert this matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 8 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1/2 & 1/2 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3/2 & -1/2 & 0 \\ 0 & 1 & 4 & -1/2 & 1/2 & 0 \\ 0 & 0 & 3 & 1/2 & -3/2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3/2 & -1/2 & 0 \\ 0 & 1 & 4 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/6 & -1/2 & 1/3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -7/6 & 5/2 & -4/3 \\ 0 & 0 & 1 & 1/6 & -1/2 & 1/3 \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ -7/6 & 5/2 & -4/3 \\ 1/6 & -1/2 & 1/3 \end{bmatrix}.$$

Now we just need to compute

$$A^{-1}(4,7,1) = \begin{bmatrix} 2 & -2 & 1 \\ -7/6 & 5/2 & -4/3 \\ 1/6 & -1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 23/2 \\ -5/2 \end{bmatrix}$$

$$f(x) = -5 + \frac{23}{2}x - \frac{5}{2}x^2.$$

4. Let U, V be vector spaces, and $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Let $L : U \to V$ be a linear map. Prove that $L(E)$ spans $L(U)$.

   Solution: Let $\mathbf{v} \in L(U)$. Then there is some $\mathbf{u} \in U$ such that $L(\mathbf{u}) = \mathbf{v}$. Since E is a basis for U, there are a_i such that $\mathbf{u} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. Then we have

$$\begin{aligned} \mathbf{v} = L(\mathbf{u}) &= L(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) \\ &= a_1 L(\mathbf{e}_1) + \cdots + a_n L(\mathbf{e}_n) \end{aligned}$$

   and thus $\mathbf{v}$ is in the span of $L(E)$. Thus $L(E)$ spans $L(U)$.

5. Let E be the standard basis for $\mathbb{R}^3$, and let $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

   (a) Find the transition matrix corresponding to the change of basis from E to F.

   (b) For each of the following vectors (expressed in the standard basis), find the coordinates with respect to F: $(3, 2, 5)$; $(1, 1, 2)$; $(2, 3, 2)$.

   Solution:

   (a) We have

$$U = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

is the transition matrix from F to E. To find the inverse of this matrix we calculate

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

so the inverse is

$$U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

This is the change of basis matrix from E to F.

(b)

$$U^{-1}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$$

$$U^{-1}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$U^{-1}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

6. Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \left\{\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$, and let $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ be two bases for $\mathbb{R}^3$.

(a) Find the transition matrix from E to F.

(b) If $\mathbf{x} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3$, find the coordinates of $\mathbf{x}$ with respect to F.

Solution:

(a) The transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition matrix from E to F is given by $B^{-1}A$. To compute B^{-1} we write

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

so

$$B^{-1}A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -5 & 0 & -1 \\ 7 & 1 & 2 \end{bmatrix}.$$

(b)

$$[\mathbf{x}]_F = B^{-1}A[\mathbf{x}]_E = \begin{bmatrix} 2 & -1 & -1 \\ -5 & 0 & -1 \\ 7 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 9 \end{bmatrix}.$$

We can check our work by computing

$$\mathbf{x} = 2\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

$$5\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix} = \mathbf{x}.$$

7. Let

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + 2z \end{bmatrix}.$$

Let A be the matrix of L with respect to the standard basis, and let B be the matrix of L with respect to the basis $F = \{(1,1,0), (1,0,1), (0,1,1)\}$.

(a) Calculate B, the matrix of L with respect to F directly.

(b) Calculuate B by finding the matrix U corresponding to a change of basis from F to the standard basis, and calculating $U^{-1}AU$.

Solution:

(a)

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

So the matrix of L with respect to F is

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(b) We have the matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and the transition matrix from F to the standard basis is

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We compute

$$\begin{aligned}
&\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 1 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]
\end{aligned}$$

so

$$U^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Then we have

$$\begin{aligned}
B = U^{-1}AU &= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.
\end{aligned}$$

This matches our earlier computation for B.

8. ($\star$) Let $T : \mathcal{P}_3(x) \to \mathcal{P}_3(x)$ be defined by $L(f(x)) = xf'(x) + f''(x)$.

(a) Find the matrix A representing T with respect to $E = \{1, x, x^2\}$.

(b) Find the matrix B representing T with respect to $F\{1, x, 1 + x^2\}$.

(c) Find the matrix S such that $B = S^{-1}AS$.

(d) If $p(x) = a_0 + a_1x + a_2(1 + x^2)$, calculate $T^n(p(x)) = T(T(\dots(T(p(x)))))$.

Solution:

(a)

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b)

$$\begin{aligned} T(1) &= 0 \to (0,0,0) \\ T(x) &= x \to (0,1,0) \\ T(1+x^2) &= 2 + 2x^2 = 2(1+x^2) \to (0,0,2) \\ B &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

(c) S here is the change of basis matrix from F to E. So we expect

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and indeed we have

$$\begin{aligned} S^{-1} &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ S^{-1}AS &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

(d) $[p]_F = (a_0, a_1, a_2)$, so

$$\begin{aligned} [T^n(p(x))]_F = B^n[p]_F &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^n \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 2^n a_2 \end{bmatrix}. \end{aligned}$$

Thus

$$T^n(p(x)) = a_1x + 2^n a_2(1 + x^2) = 2^n a_2 + a_1 x + 2^n a_2 x^2.$$