Math 214 Test 1
Practice Problem Solutions

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

These are partial solutions. I don't know how many I'll get up, but I'll try to improve it as I can.

Which of the following are vector spaces? Prove or disprove your answer, potentially using the subspace theorem

1. $\{(a, b, c, d) : a - b = c - d\}$
2. $\{(a, b, c, d) : a + b + c = d\}$
3. $\{(a, b, c) : a^2 = bc\}$
4. $\{(a, b, c, d) : 5a - 3b = 2c - 2d\}$
5. $\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_2 = 2\}$
6. $\{f(x) : f(0) = 5\}$
7. $\{f(x) : f(5) = 0\}$

Solution:

1. Yes
2. Yes
3. No: $(0, 1, 0)$ and $(0, 0, 1)$ are in the set, but $(0, 1, 0) + (0, 0, 1) = (0, 1, 1)$ is not.
4. Yes
5. No: $2x^2$ is in the set, but $2 \cdot 2x^2 = 4x^2$ is not.
6. No: the constant function $f(x) = 5$ is in the set, but $(2f)(x) = 10$ so $2f$ is not in the set.
7. Yes.

Write u as a linear combination of vectors in S, or prove you cannot

1. $\mathbf{u} = (5, 2, 1)$, $S = \{(1, 2, 3), (3, 1, 1)\}$

   Solution: Not possible
2. $\mathbf{u} = (2, 3, 2)$, $S = \{(1, 2, 3), (3, 4, 1)\}$

   Solution: $\mathbf{u} = \frac{1}{2}(1, 2, 3) + \frac{1}{2}(3, 4, 1)$.
3. $\mathbf{u} = x^3 - x + 1$, $S = \{1 + x, 3 + x^2, 3x^2 + x^3\}$

   Solution: Not possible
4. $\mathbf{u} = x^3 + 4x^2 + 2x + 5$, $S = \{1 + x, 3 + x^2, 3x^2 + x^3\}$

   Solution: $2(1 + x) + (3 + x^2) + (3x^2 + x^3) = 5 + 2x + 2x^3 + x^3$.

For each of the following sets, check:

- Does it span the (implicitly given) vector space?
- Is it linearly independent?
- Is it a basis?

1. $S = \{(2,1,-2),(-2,-1,2),(4,2,-4)\}$
2. $S = \{(2,1,-2),(3,2,-2),(2,2,0)\}$
3. $S = \{(1,0,0,1),(0,1,0,0),(2,3,0,2)\}$
4. $S = \{(1,5,2),(3,1,4),(-1,3,7),(2,8,1)\}$
5. $S = \{1+x^2, 1+x^3, x-x^2, 5+x^2-4x^3\}$
6. $S = \{1+2x, x+2x^2, x^2+2x^3, 2+x^3\}$

Solution:

1. No, no, no.
2.
3. No, no, no.
4.
5.
6. Yes, yes, yes.

Bases

1. Find a basis for $\mathbb{R}^3$ containing $(-1,3,2)$ and $(5,4,1)$.

   Solution: It looks like $(1,0,0)$ is not in the span of $\{(-1,3,2),(5,4,1)\}$, so we test:

   $$\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}-1\\3\\2\end{bmatrix} + b\begin{bmatrix}5\\4\\1\end{bmatrix} = \begin{bmatrix}5b-a\\3a+4b\\2a+b\end{bmatrix}$$

   giving

   $$5b-a=1 \qquad 3a+4b=0 \qquad 2a+b=0.$$

   This gives us $b=-2a$, so $3a-8a=0$ implying $a=0$ and $b=0$, and then we have $0=1$ a contradiction, so $(1,0,0)$ is indeed not in the span. Thus basis padding tells us that $\{(-1,3,2),(5,4,1),(1,0,0\}$ is a basis.

2. Find a basis for $\mathbb{R}^3$ containing $(7,1,-3)$ and $(1,1,1)$.

   It looks like $(1,0,0)$ is not in the span of $\{(7,1,-3),(1,1,1)\}$, so we test:

   $$\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}7\\1\\-3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}7a+b\\a+b\\b-3a\end{bmatrix}$$

   giving

   $$7a+b=1 \qquad a+b=0 \qquad b-3a=0.$$

   This gives us $a=-b$ so $b+3b=0$ implies $b=0$ and thus $a=0$, so we have $0=1$, a contradiction. Thus $(1,0,0)$ is not in the span, so by basis padding $\{(7,1,-3),(1,1,1),(1,0,0)\}$ is a basis for $\mathbb{R}^3$.

3. Find a basis for $\mathbb{R}^4$ containing $(1,2,3,4), (1,1,1,1)$, and $(0,0,1,1)$.
4. Find a basis for $\mathcal{P}_3(x)$ containing $1+3x^3, x^2-x, 6-2x$.

   Solution: We guess that you can't get 1 in the span of this set. We check by supposing

$$1 = a(1+3x^3) + b(x^2-x) + c(6-2x) = a + 6c - (b+2c)x + bx^2 + 3ax^3)$$

   and thus we get the equations

$$1 = a + 6c \qquad 0 = -b - 2c0 \qquad = b0 \qquad = 3a.$$

   this gives that $0 = a, 0 = b$ from the last two equations, and thus $0 = c$ from the second equation, which gives a contradiction $1 = 0$ from the first equation. Thus 1 is indeed not in the span.

   Thus $\{1, 1+3x^2, x^2-x, 6-2x\}$ is a basis for $\mathcal{P}_3(x)$, since it is linearly independent and the dimension of $\mathcal{P}_3(x)$ is 4.

5. Find a basis for $\mathbb{R}^3$ that is a subset of $\{(1,1,1), (2,4,6), (7,-1,2), (2,5,-2), (3,-6,4)\}$.
6. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(1,3), (2,4), (1,1)\}$.
7. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(-1,4), (7,-2), (3,6)\}$.
8. Find a basis for $\mathcal{P}_2(x)$ that is a subset of $\{1+x, 3+x^2, 4+3x+2x^2, x^2-7x\}$.

Proofs

1. Suppose U, W are subspaces of some vector space V. Prove that the set $U + W = \{\mathbf{u}+\mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$ is a subspace of V.

   Bonus: what is the space $U + U$?

   Solution: We know that $\mathbf{0} \in U$ and $\mathbf{0} \in W$, so $\mathbf{0}+\mathbf{0} = \mathbf{0} \in U + W$.

   Suppose $\mathbf{u}_1 + \mathbf{w}_1, \mathbf{u}_2 + \mathbf{w}_2 \in U + W$. Then $(\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \in U + W$ since $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{w}_1 + \mathbf{w}_2 \in W$ by closure under addition of vector spaces.

   Suppose $\mathbf{u} + \mathbf{w} \in U + W$, and $r \in \mathbb{R}$. Then $r(\mathbf{u}+\mathbf{w}) = r\mathbf{u} + r\mathbf{w} \in U + W$ since $r\mathbf{u} \in U$ and $r\mathbf{w} \in W$ by closure under scalar multiplication of vector spaces.

   Thus by the subspace theorem, $U + W$ is a subspace.

   Bonus: $U + U = U$, essentially because U is closed under addition.

2. Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq V$ is linearly independent. Show that $T = \{\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_n - \mathbf{v}_{n-1}\}$ is linearly independent.

   Solution: This is similar to problem 4 on homework 3.

   We want to prove T is linearly independent, so suppose we have some scalars a_i such that

$$\begin{aligned} 0 &= a_1\mathbf{v}_1 + a_2(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + a_n(\mathbf{v}_n - \mathbf{v}_{n-1}) \\ &= (a_1 - a_2)\mathbf{v}_1 + (a_2 - a_3)\mathbf{v}_2 + \cdots + (a_{n-1} - a_n)\mathbf{v}_{n-1} + a_n\mathbf{v}_n. \end{aligned}$$

   Then since S is linearly independent, we know all the coefficients in that last equation must be zero; thus we have

$$\begin{aligned} 0 &= a_1 - a_2 & 0 &= a_2 - a_3 \\ &\vdots & &\vdots \\ 0 &= a_{n-1} - a_n & 0 &= a_n \end{aligned}$$

and rearranging these equations gives

$$\begin{aligned} a_1 &= a_2 & a_2 &= a_3 \\ &\vdots & &\vdots \\ a_{n-1} &= a_n & a_n &= 0 \end{aligned}$$

And thus $a_i = 0$ for every a_i. Thus by definition T is linearly independent.

Bonus to stretch your brain

1. Find a subset $U \subset \mathbb{R}^2$ that is closed under scalar multiplication but is not a subspace.

   Solution: One possible example is $\{(x, y) : x^2 = y^2\}$. If $x^2 = y^2$ then $(ax)^2 = (ay)^2$ so it's closed under scalar multiplication. But $(1, 1) + (1, -1) = (2, 0)$ is not in this subset, even though $(1, 1)$ and $(1, -1)$ are.

2. Find a subset $U \subset \mathbb{R}^2$ that is closed under addition but is not a subspace.

   Solution: One possible answer is $\mathbb{Z}^2 = \{(a, b) : a, b \in \mathbb{Z}\}$ the set of ordered pairs of integers. If $(a, b), (x, y) \in \mathbb{Z}^2$ then $(a + x, b + y) \in \mathbb{Z}^2$. But we see that while $(1, 0) \in \mathbb{Z}^2$, $.5 \cdot (1, 0) = (.5, 0)$ is not.

3. Can you find a basis for $\mathcal{P}_3(x)$ such that no element of the basis has degree 3?

   Solution: No, because then no element will have a nonzero coefficient for x^3 and thus x^3 is not in the span of the set.

4. Can you find a basis for $\mathcal{P}_3(x)$ such that no element of the basis has degree 2?

   Solution: Yes; an example is $\{1, x, x^2 + x^3, x^3\}$. This is different from the previous problem since I can include the nonzero x^2 term in a polynomial whose degree is not 2.