

Math 214 Test 2

Practice Problem Solutions

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

These are partial solutions. I should be able to finish the solutions tomorrow.

There are almost certainly a couple typos in here. If you think you've found a typo: first download the newest version from the website; I might have found and fixed it already. If not, send me an email or come see me.

Do the following matrix multiplication computations.

1.

$$\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} =$$

Solution:

$$\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & 25 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix} =$$

Solution:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 10 & 14 \\ 6 & 2 & -5 \\ 5 & 4 & 11 \end{bmatrix}$$

3.

$$\begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 0 & 1 \\ 4 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} =$$

Solution:

$$\begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 0 & 1 \\ 4 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -7 & -4 & 4 \\ -3 & 6 & 10 \\ -4 & 6 & 18 \end{bmatrix}$$

4.

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 2 & 3 & 8 \end{bmatrix} =$$

Solution: Undefined, since the dimensions are wrong.

For each of the following matrices, find:

- (a) The reduced row echelon form.
- (b) A basis for the row space.
- (c) A basis for the column space.
- (d) The rank.
- (e) A basis for the nullspace.
- (f) The nullity.

1.
$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: The reduced row echelon form is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
, so:

- The row space has basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
- The column space has basis $\{(3, -1, 1), (1, 4, 1), (2, 2, 1)\}$.
- The rank is 3.
- The nullspace is trivial and so has basis $\{\}$.
- The nullity is zero.

2.
$$\begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$$

Solution: The reduced row echelon form is
$$\begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$$
, so

- The row space has basis $\{(1, 0, -1/6), (0, 1, 1/6)\}$.
- The column space has basis $\{(-2, -5, 3), (4, 1, 3)\}$.
- The rank is 2.
- The nullspace is $\{\alpha/6, -\alpha/6, \alpha\}$ and so has basis $\{(1, -1, 6)\}$ or $\{1/6, -1/6, 1\}$.
- The nullity is one.

3.
$$\begin{bmatrix} 6 & 2 & 3 & 1 \\ 1 & 5 & 2 & -2 \\ 4 & -4 & 1 & 3 \end{bmatrix}$$

Solution: The reduced row echelon form is
$$\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & -1/5 \end{bmatrix}$$
, so

- The row space has basis $\{(1, 0, 0, 2/5), (0, 1, 0, -2/5), (0, 0, 1, -1/5)\}$.

- The column space has basis $\{(6, 1, 4), (2, 5, -4), (3, 2, 1)\}$.
- The rank is 3.
- The nullspace is $\{(-2\alpha/5, 2\alpha/5, \alpha/5, \alpha)\}$ and so has basis $\{(-2, 2, 1, 5)\}$ or $\{(-2/5, 2/5, 1/5, 1)\}$.
- The nullity is one.

4.
$$\begin{bmatrix} -1 & 3 & 4 \\ 2 & 5 & 2 \\ 0 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix}$$

Solution: The reduced row echelon form is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
, so

- The row space has basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
- The column space has basis $\{(-1, 2, 0, 4), (3, 5, 1, 1), (4, 2, 3, -2)\}$.
- The rank is 3.
- The nullspace is trivial and so has basis $\{\}$.
- The nullity is zero.

Find the rank and nullity of the following matrices

You shouldn't need to do any actual computations here. (Hint: Rank-Nullity theorem).

1.
$$\begin{bmatrix} 1 & 1 & 1 & 32 & 217 & 53 - e & 3^3 \\ 0 & 1 & 0 & 512 & 256 & 128 & 64 \\ 0 & 1 & 1 & 12345 & 4^{4^4} & 2 & 0 \end{bmatrix}$$

Solution: The rows are linearly independent, so the rank is 3. By the rank-nullity theorem, the nullity is 4.

2.
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Solution: The rows are all scalar multiples of each other, so the rank is 1. By the rank-nullity theorem, the nullity is 3.

3.
$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

Solution: The third row is a combination of the first two, so the rank is 2. By the rank-nullity theorem, the nullity is 2.

4.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: There are three linearly independent rows, since the zero rows don't contribute, so the rank is 3. The nullity is 0.

Find the inverses of the following matrices, or show they are not invertible.

1. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Solution: We form the augmented matrix $\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array} \right]$ and row-reduce to get $\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array} \right]$.

Thus the inverse is $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

Solution: We notice that the two rows are linearly dependent, so the rank is 1 and the nullity is 1, and the matrix has no inverse.

Alternatively, we form the augmented matrix $\left[\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 6 & 2 & 1 & 0 \end{array} \right]$ and row-reduce to get $\left[\begin{array}{cc|cc} 1 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & -2 \end{array} \right]$.

Thus the matrix has no inverse.

3. $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 1 & -4 & 3 \end{bmatrix}$

Solution: We form the augmented matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 1 & -4 & 3 & 0 & 0 & 1 \end{array} \right]$ and row-reduce to get

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/27 & 8/27 & 4/27 \\ 0 & 1 & 0 & 5/27 & 1/54 & -13/54 \\ 0 & 0 & 1 & 7/27 & -2/27 & -1/27 \end{array} \right]$. Thus the inverse is $\begin{bmatrix} -1/27 & 8/27 & 4/27 \\ 5/27 & 1/54 & -13/54 \\ 7/27 & -2/27 & -1/27 \end{bmatrix}$.

4. $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 7 & 1 & 5 \end{bmatrix}$

Solution: We may notice that the third row is the sum of the first two, and thus the matrix has rank 2 and nullity 1, so is not invertible.

Alternatively, we form the augmented matrix $\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 & 1 & 0 \\ 7 & 1 & 5 & 0 & 0 & 1 \end{array} \right]$ and row-reduce to get

$\left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 0 & 1/5 & 0 \\ 0 & 1 & 11/5 & 0 & -7/5 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array} \right]$. Thus there is no inverse.

For each of the following functions

- Identify the domain and codomain.
- Determine whether it is a linear transformation.
- Prove your answer from part (b).
- If it is, find a matrix with respect to the standard basis of \mathbb{R}^n .
- If it is linear, find the kernel and image.

1. $f(x, y, z) = (3x^2, x + y, 2z - y)$.

Solution: Domain and codomain are \mathbb{R}^3 .

No, because $f(1, 0, 0) = (3, 1, 0)$ but $f(2, 0, 0) = (12, 2, 0) \neq 2(3, 1, 0)$.

2. $f(x, y, z) = (5x + y, z - 3x, y + z)$.

Solution: Domain and codomain are \mathbb{R}^3 .

$$\begin{aligned} f(r(x, y, z)) &= (5rx + ry, rz - 3rx, ry + rz) = r(5x + y, z - 3x, y + z) = rf(x, y, z) \\ f((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= f(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (5(x_1 + x_2) + (y_1 + y_2), (z_1 + z_2) - 3(x_1 + x_2), (y_1 + y_2) + (z_1 + z_2)) \\ &= (5x_1 + y_1, z_1 - 3x_1, y_1 + z_1) + (5x_2 + y_2, z_2 - 3x_2, y_2 + z_2) \\ &= f(x_1, y_1, z_1) + f(x_2, y_2, z_2). \end{aligned}$$

Thus it is linear. We have

$$f(\mathbf{e}_1) = (5, -3, 0)$$

$$f(\mathbf{e}_2) = (1, 0, 1)$$

$$f(\mathbf{e}_3) = (0, 1, 1)$$

$$A = \begin{bmatrix} 5 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

To find the kernel and image, we row reduce the matrix, getting

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which tells us that the kernel is trivial, and the image is $\text{span}\{(5, -3, 0), (1, 0, 1), (0, 1, 1)\}$. Since this is three-dimensional, the image must be \mathbb{R}^3 .

3. $f(x, y, z) = (x + y, z + y)$.

Solution: The domain is \mathbb{R}^3 and the codomain is \mathbb{R}^2 .

The function is a linear transformation, since:

$$\begin{aligned} f(r(x, y, z)) &= (rx + ry, rz + ry) = r(x + y, z + y) = rf(x, y, z) \\ f((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= ((x_1 + x_2) + (y_1 + y_2), (z_1 + z_2) + (y_1 + y_2)) \\ &= (x_1 + y_1, z_1 + y_1) + (x_2 + y_2, z_2 + y_2) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2). \end{aligned}$$

We compute

$$f(\mathbf{e}_1) = (1, 0)$$

$$f(\mathbf{e}_2) = (1, 1)$$

$$f(\mathbf{e}_3) = (0, 1)$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The row reduced echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ so the kernel is $x = z, y = -z$, or $\{(\alpha, -\alpha, \alpha)\} = \{\alpha(1, -1, 1)\}$. Thus $\{(1, -1, 1)\}$ is a basis for the kernel.

The image is spanned by $(1, 0)$ and $(1, 1)$ since these are the columns corresponding to leading ones. Thus $\{(1, 0), (1, 1)\}$ is a basis for the image, which must be all of \mathbb{R}^2 .

4. $f(x, y) = (x + y, x - y, 1)$.

Solution: The domain is \mathbb{R}^2 and the codomain is \mathbb{R}^3 .

The function is not linear, since $f(0, 0) = (0, 0, 1) \neq \mathbf{0}$. Alternately, we can see that, say, $f(1, 0) = (1, 1, 1)$ and $f(0, 1) = (1, -1, 1)$, and $f(1, 1) = (2, 0, 1) \neq (1, 1, 1) + (1, -1, 1)$.

For each of the following functions

- (a) Determine whether it is a linear transformation.
- (b) Prove your answer from part (a).
- (c) If it is, find a matrix with respect to the given bases.
- (d) If it is linear, find the kernel and image.

1. $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $L(x, y) = (x, y, x + y)$, with respect to $E = \{(1, 1), (1, -1)\}$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

Solution: L is a linear transformation, since

$$\begin{aligned}L(r(x, y)) &= (rx, ry, rx + ry) = r(x, y, x + y) = rL(x, y) \\L((x_1, y_1) + (x_2, y_2)) &= (x_1 + x_2, y_1 + y_2, x_1 + x_2 + y_1 + y_2) \\&= (x_1, y_1, x_1 + y_1) + (x_2, y_2, x_2 + y_2) = L(x_1, y_1) + L(x_2, y_2).\end{aligned}$$

To find the matrix, we compute

$$\begin{aligned}L(1, 1) &= (1, 1, 2) = -(1, 1, 0) + 2(1, 1, 1) \rightarrow (0, -1, 2) \\L(1, -1) &= (1, -1, 0) = 2(1, 0, 0) - 1(1, 1, 0) \rightarrow (2, -1, 0) \\A &= \begin{bmatrix} 0 & 2 \\ -1 & -1 \\ 2 & 0 \end{bmatrix}.\end{aligned}$$

To find the kernel and image, we row-reduce the matrix to get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus the kernel is trivial, and the image is the span of $\{(0, -1, 2), (2, -1, 0)\}$ in F ; in other words, the span of $\{(1, 1, 2), (1, -1, 0)\}$. (Which is of course the span of the image of the basis, as we proved on your last homework!)

2. $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $L(x, y, z) = (x + y + z, x - y)$, with respect to $E = \{(1, 1, 1), (1, -1, 0), (0, 0, 1)\}$ and $F = \{(3, 0), (0, 2)\}$.

Solution: L is a linear transformation, since

$$\begin{aligned}L(r(x, y, z)) &= (rx + ry + rz, rx - ry) = r(x + y + z, x - y) = rL(x, y, z) \\L((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= (x_1 + x_2 + y_1 + y_2 + z_1 + z_2, x_1 + x_2 - (y_1 + y_2)) \\&= (x_1 + y_1 + z_1, x_1 - y_1) + (x_2 + y_2 + z_2, x_2 - y_2) \\&= L(x_1, y_1, z_1) + L(x_2, y_2, z_2).\end{aligned}$$

To find the matrix, we compute

$$\begin{aligned}L(1, 1, 1) &= (3, 0) \rightarrow (1, 0) \\L(1, -1, 0) &= (0, 2) \rightarrow (0, 1) \\L(0, 0, 1) &= (1, 0) \rightarrow (1/3, 0) \\A &= \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}.\end{aligned}$$

To find the kernel, we see the matrix is already row-reduced. Thus $N(A) = \{(-\alpha/3, 0, \alpha)\} = \{\alpha(-1/3, 0, 1)\}$, so a basis for the kernel is $\{(-1, 0, 3)\}$ or $\{-1/3, 0, 1\}$ in E . Thus a basis for the kernel is $\{(-1, -1, 2)\}$.

The image is the span of $\{(1, 0), (0, 1)\}$ in F , and thus the span of $\{(3, 0), (0, 2)\}$. Thus $\{(3, 0), (0, 2)\}$ is a basis for the image.

3. $L : \mathcal{P}_3(x) \rightarrow \mathbb{R}^2$ given by $L(f(x)) = (f(1), f(2))$, with $E = \{1, x, x^2, x^3\}$ and $F = \{(1, 0), (0, 1)\}$.

Solution: This is a linear transformation, since

$$\begin{aligned} L(rf(x)) &= (rf(1), rf(2)) = r(f(1), f(2)) = rL(f(x)) \\ L((f+g)(x)) &= ((f+g)(1), (f+g)(2)) = (f(1)+g(1), f(2)+g(2)) \\ &= (f(1), f(2)) + (g(1), g(2)) = L(f(x)) + L(g(x)). \end{aligned}$$

To compute the matrix, we have

$$\begin{aligned} L(1) &= (1, 1) \\ L(x) &= (1, 2) \\ L(x^2) &= (1, 4) \\ L(x^3) &= (1, 8) \\ A &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}. \end{aligned}$$

Row reducing this matrix gives

$$\begin{bmatrix} 1 & 0 & -2 & -6 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

so the kernel is given by $\{(2\alpha + 6\beta, -3\alpha - 7\beta, \alpha, \beta)\} = \{\alpha(2, -3, 1, 0) + \beta(6, -7, 0, 1)\}$ in E , and thus a basis is $\{2 - 3x + x^2, 6 - 7x + x^3\}$.

The image has basis $\{(1, 1), (1, 2)\}$ and thus is all of \mathbb{R}^2 .

4. $L : \mathcal{P}_2(x) \rightarrow \mathcal{P}_3(x)$ given by $L(f(x)) = \int_0^x f(t) dt$, with $E = \{1, x, x^2\}$ and $F = \{1, x, x^2, x^3\}$.

Solution: L is linear, since from calculus we know that $\int_0^x rf(t) dt = r \int_0^x f(t) dt$, and $\int_0^x f(t) + g(t) dt = \int_0^x f(t) dt + \int_0^x g(t) dt$.

To find the matrix we compute

$$\begin{aligned} L(1) &= \int_0^x 1 dt = x \rightarrow (0, 1, 0, 0) \\ L(x) &= \int_0^x t dt = x^2/2 \rightarrow (0, 0, 1/2, 0) \\ L(x^2) &= \int_0^x t^2 dt = x^3/3 \rightarrow (0, 0, 0, 1/3) \\ A &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \end{aligned}$$

The reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the kernel is trivial, and the image is the span of $\{(0, 1, 0, 0), (0, 0, 1/2, 0), (0, 0, 0, 1/3)\}$ in F , or in other words the span of $\{(x, x^2/2, x^3/3)\}$. Thus the image is all cubic polynomials with zero constant term.

5. $L : \mathbb{R}^3 \rightarrow \mathcal{P}_3(x)$ given by $L(a, b, c) = (a + b + c) + ax + bx^2 + cx^3$, with $E = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and $F = \{1, 1 + x, 1 + x^2, 1 + x^3\}$.

Solution: The function is linear, since

$$\begin{aligned} L(r(a, b, c)) &= (ra + rb + rc) + rax + rbx^2 + rcx^3 = rL(a, b, c) \\ L((a_1, b_1, c_1) + (a_2, b_2, c_2)) &= (a_1 + a_2 + b_1 + b_2 + c_1 + c_2) + (a_1 + a_2)x + (b_1 + b_2)x^2 + (c_1 + c_2)x^3 \\ &= (a_1 + b_1 + c_1) + a_1x + b_1x^2 + c_1x^3 + (a_2 + b_2 + c_2) + a_2x + b_2x^2 + c_2x^3 \\ &= L(a_1, b_1, c_1) + L(a_2, b_2, c_2). \end{aligned}$$

To find the matrix we compute

$$L(1, 1, 1) = 3 + x + x^2 + x^3 \rightarrow (0, 1, 1, 1)$$

$$L(1, 1, 0) = 2 + x + x^2 \rightarrow (0, 1, 1, 0)$$

$$L(1, 0, 0) = 1 + x \rightarrow (0, 1, 0, 0)$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The row-reduced form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the kernel is trivial. The image is the span of $\{3 + x + x^2 + x^3, 2 + x + x^2, 1 + x\}$.

6. The function $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by rotating 90 degrees counterclockwise around the z axis, and then 135 degrees counterclockwise around the x axis.

Solution: This function is linear since rotations are linear: stretching then rotating is the same as rotating then stretching, and adding then rotating is the same as rotating then adding.

To find the matrix we compute:

$$R(1, 0, 0) = (0, -\sqrt{2}, \sqrt{2})$$

$$R(0, 1, 0) = (-1, 0, 0)$$

$$R(0, 0, 1) = (0, -\sqrt{2}, -\sqrt{2})$$

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -\sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & -\sqrt{2} \end{bmatrix}.$$

It's not too hard to see that the rows are linearly independent, so the rank is 3 and thus a basis for the columnspace is $\{(0, -\sqrt{2}, \sqrt{2}), (-1, 0, 0), (0, -\sqrt{2}, -\sqrt{2})\}$, or $\{(0, -1, 1), (1, 0, 0), (0, 1, 1)\}$. Thus the columnspace spans \mathbb{R}^3 . (This makes sense since we know we can land anywhere by rotating). Since the rank is 3, the kernel is trivial, which again makes geometric sense.

Proofs

1. Let $L : U \rightarrow V$ be a linear transformation of vector spaces, with bases E and F . Let A be the matrix of L with respect to E and F . Prove that $\mathbf{v} \in \ker(L)$ if and only if $[\mathbf{v}]_E$ is in the nullspace of A .

Solution: Recall that by definition, $A[\mathbf{v}]_E = [L(\mathbf{v})]_F$. So $\mathbf{v} \in \ker(L)$ if and only if $L(\mathbf{v}) = \mathbf{0}$ if and only if $[L(\mathbf{v})]_F = \mathbf{0}$ if and only if $A[\mathbf{v}]_E = \mathbf{0}$ if and only if $[\mathbf{v}]_E \in N(A)$ by definition.

2. Let $L : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ such that $\ker(L) = \{(x, y, z, w) : x = y, z = w\}$. Prove that the image of L is \mathbb{R}^2 .

Solution: Let A be the matrix of L ; then $A \in M_2 \times 4$. We see that the kernel of L is $\{\alpha(1, 1, 0, 0) + \beta(0, 0, 1, 1)\}$ and thus two-dimensional. Thus the nullity of A is 2, and by the rank-nullity theorem, the rank of A is $4 - 2 = 2$.

Thus the column space of A is two-dimensional. But the column space of A is the image of L , so the image of L is a two-dimensional subspace of \mathbb{R}^2 . The only such subspace is \mathbb{R}^2 itself.

3. Let $A \in M_{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that if $N(A) = \{\mathbf{0}\}$ then the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

Solution: There are two approaches.

One is to recall the theorem from class: let's assume the equation $A\mathbf{x} = \mathbf{b}$ has a solution \mathbf{x}_0 . Then the set of solutions looks like $\{\mathbf{x}_0 + \mathbf{u} : \mathbf{u} \in N(A)\}$. But since $N(A) = \{\mathbf{0}\}$ then the set of solutions is $\{\mathbf{x}_0 + \mathbf{0}\} = \{\mathbf{x}_0\}$ and thus there is only one solution.

If we don't remember that theorem, we can prove it directly. Suppose $A\mathbf{x} = \mathbf{b} = A\mathbf{y}$. Then $A\mathbf{x} - A\mathbf{y} = \mathbf{0}$, so $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, and thus $\mathbf{x} - \mathbf{y} \in N(A)$.

But $N(A) = \{\mathbf{0}\}$ so this implies that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, and thus $\mathbf{x} = \mathbf{y}$. Thus any two solutions are equal, so there is at most one solution.

4. Let U, V be 2-dimensional subspaces of \mathbb{R}^3 . On your first test you showed that the set $U \cap V = \{\mathbf{u} : \mathbf{u} \in U, \mathbf{u} \in V\}$ of vectors in both U and V is a subspace of \mathbb{R}^3 . Prove that $\dim(U \cap V) \neq 0$. (Hint: let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for U and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for V . What can you say about $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$?)

Solution: The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ is a four-element subset of \mathbb{R}^3 , and thus is linearly dependent. Thus there are some a_i not all zero such that

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{v}_1 + a_4\mathbf{v}_2 = \mathbf{0}.$$

If $a_1 = a_2 = 0$ then $a_3\mathbf{v}_1 + a_4\mathbf{v}_2 = \mathbf{0}$, and since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis it is linearly independent, so $a_3 = a_4 = 0$, a contradiction. So we can assume without loss of generality that $a_1 \neq 0$.

Then we have $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 = -a_3\mathbf{v}_1 - a_4\mathbf{v}_2$. The left-hand side is an element of U , and is non-zero since $a_1 \neq 0$ and $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent. The right-hand side is an element of V . Thus there is some non-zero element of U that is also an element of V , so $U \cap V \neq \{\mathbf{0}\}$.