Math 214 Test 3
Practice Problem Solutions

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

These are partial solutions. I should be able to finish the solutions tomorrow.

There are almost certainly a couple typos in here. If you think you've found a typo: first download the newest version from the website; I might have found and fixed it already. If not, send me an email or come see me.

Proofs

1. Let Q be the subspace of $\mathcal{P}(x)$ consisting of polynomials with zero constant term. Prove that the function $D : Q \to \mathcal{P}(x)$ given by the derivative is an isomorphism.

   Solution: We know that D is linear, so we just need to prove that it is one-to-one and onto. Suppose $D(a_1 x + \cdots + a_n x^n) = 0$. then we have $0 = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}$ and thus $a_1 = 2a_2 = \cdots = na_n = 0$ so $a_1 = a_2 = \cdots = a_n = 0$. Thus $D(f) = 0$ implies $f = 0$, so $\ker(D) = \{0\}$ and thus D is one-to-one.

   Conversely, let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}(x)$. Then let $g(x) = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1} \in Q$, and we see that $D(g) = f$. Thus D is onto.

   Consequently we see that D is one-to-one and onto, thus it is an isomorphism by definition.

2. Let $U = \operatorname{span}\{x, \sin(x), \cos(x), x^5, 1\}$. Find an isomorphism between U and $\mathbb{R}^5$.

   Solution: Define L by

   $$L(a_1 x + a_2 \sin(x) + a_3 \cos(x) + a_4 x^5 + a_5) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

   Then L takes a basis to a basis, and thus is an isomorphism.

3. Suppose V is a vector space and $L : V \to \mathbb{R}^5$ is surjective and $\dim \ker(L) = 2$. What can you say about V?

   Solution: By the rank-nullity theorem, $\dim V = \dim \ker(L) + \dim L(V)$. We know that $L(V) = \mathbb{R}^5$ so $\dim L(V) = 5$, and $\dim \ker(L) = 2$. Thus $\dim V = 7$.

4. Suppose $T : \mathbb{R}^5 \to \mathcal{P}_4(x)$ and $\dim \ker(T) = 1$. What can you say about $T(\mathbb{R}^5)$?

   Solution: By the Rank-Nullity Theorem, we know that $\dim \mathbb{R}^5 = \dim \ker(T) + \dim T(\mathbb{R}^5)$, and thus $5 = 1 + \dim T(\mathbb{R}^5)$, so $\dim T(\mathbb{R}^5)$ is four-dimensional. Thus T is not surjective since $\mathcal{P}_4(x)$ is five-dimensional.

Determine whether the following operators are invertible

1. $L(x, y, z) = (x, x + y, x + z)$

   Solution: Suppose $L(x, y, z) = (0, 0, 0)$. Then we have $x = 0, x + y = 0, x + z = 0$ and thus $x + y + z = 0$. Thus $\ker(L) = \{\mathbf{0}\}$, so L is injective. Since the domain and codomain have the same dimension, it is invertible.

2. $L(x, y, z) = (x + y, y + z, x + z)$

   Solution: The matrix of L is

   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

   which is rank 3 and nullity 0, and thus is invertible. So L is also invertible.

3. $L(x, y, z, w) = (x + y, y + z, z + w, w + x)$

   Solution: We see that $L(1, -1, 1, -1) = (0, 0, 0, 0)$. The kernel is non-trivial, so the operator is not invertible.

4. $L(x, y) = (x, y, 3x + 2y)$

   Solution: The domain and codomain have different dimensions, so the operator is not invertible. In particular, the operator is not surjective, since $(1, 1, 1)$ is not in the image; if the first two coordinates are both 1, the third must be 5.

5. $L(x, y, z) = (x + y, z + y)$

   Solution: The domain and codomain have different dimensions, so the operator is not invertible. In particular, We see that $L(1, -1, 1) = (0, 0)$ so the kernel is nontrivial and thus the operator is not injective.

6. $L : \mathcal{P}_3(x) \to \mathbb{R}^3$ given by $L(f) = (f(1), f(2), f(3))$.

   Solution: Again, the domain and codomain have different dimensions, since $\dim \mathcal{P}_3(x) = 4$. Thus L cannot be invertible. In particular, we see that $(x - 1)(x - 2)(x - 3) \in \ker(L)$.

Find inverses for the following operators

1. $L(x, y) = (3x + y, 2x - 4y)$

   Solution: The matrix of L is

   $$A = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$$

   so we compute

   $$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & -4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 2 & -4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & -14/3 & -2/3 & 1 \end{array}\right]$$
   $$\to \left[\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1 & 1/7 & -3/14 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2/7 & 1/14 \\ 0 & 1 & 1/7 & -3/14 \end{array}\right]$$

   so

   $$L^{-1}(a, b) = \begin{bmatrix} 2a/7 + b/14 \\ a/7 - 3b/14 \end{bmatrix}.$$

2. $L(x, y, z) = (3x + y, 2y + 2z, x - z)$

Solution: The matrix of L is

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

so we compute

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & -3 \\ 0 & 2 & 2 & 0 & 1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & -3 \\ 0 & 0 & -4 & -2 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1/2 & -1/4 & -3/2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/4 & -1/2 \\ 0 & 1 & 0 & -1/2 & 3/4 & 3/2 \\ 0 & 0 & 1 & 1/2 & -1/4 & -3/2 \end{array}\right]$$

so we have

$$L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a/2 - b/4 - c/2 \\ -a/2 + 3b/4 + 3c/2 \\ a/2 - b/4 - 3c/2 \end{bmatrix}.$$

3. $L(x, y, z) = (x + y + z, 2x - 2y, z)$

Solution: The matrix of L is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so we compute

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & -4 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1/2 & -1/4 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/4 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/4 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

so

$$L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a/2 + b/4 - c/2 \\ a/2 - b/4 + c/2 \\ c \end{bmatrix}.$$

4. $L : \mathcal{P}_2(x) \to \mathbb{R}^3$ given by $L(f) = (f(0), f(1), f(2))$.

Solution: The matrix of L is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

so we compute

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 2 & -1/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right]$$

so we have

$$L^{-1}(a,b,c) = a + (-3a/2 + 2b - c/2)x + (a/2 - b + c/2)x^2.$$

Find the transition matrices between the following bases

1. The standard basis and

$$F = \left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \right\}$$

Solution: The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 6 \\ 1 & 4 & 3 \end{bmatrix}.$$

The transition matrix from the standard basis to F is

$$A^{-1} = \begin{bmatrix} 3/14 & 1/35 & -9/70 \\ 0 & -1/5 & 2/5 \\ -1/14 & 9/35 & -11/70 \end{bmatrix}.$$

2. The standard basis and

$$F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Solution: The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The transition matrix from the standard basis to F is

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}.$$

3.

$$E = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution:

The transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the transition matrix from the standard basis to F is

$$B^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

So the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & 1/2 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}$$

and the transition matrix from F to E is

$$(B^{-1}A)^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}.$$

4.

$$E = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution: The transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition matrix from the standard basis to F is then

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to E is

$$(B^{-1}A)^{-1} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 2 & 1/2 \\ -3/2 & -5 & -1/2 \end{bmatrix}.$$

Write the given element in the given basis

1. Write $(3,1,4)$ in the basis $F = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$.

   Solution: The transition matrix from F to the standard basis is

   $$A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

   so the transition matrix from the standard basis to F is the inverse inverse

   $$A^{-1} = \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}.$$

   Thus

   $$[(3,1,4)]_F = \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}3\\1\\4\end{bmatrix} = \begin{bmatrix}2\\-3\\4\end{bmatrix}.$$

2. Write $(2,7,1)$ in the basis $F = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$.

   Solution: The transition matrix from F to the standard basis is

   $$A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}$$

   so the transition matrix from the standard basis to F is

   $$A^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1 & 1\\1 & 1 & -1\\-1 & 1 & 1\end{bmatrix}$$

   so

   $$[(2,7,1)]_F = \frac{1}{2}\begin{bmatrix}1 & -1 & 1\\1 & 1 & -1\\-1 & 1 & 1\end{bmatrix} \begin{bmatrix}2\\7\\1\end{bmatrix} = \begin{bmatrix}-2\\4\\3\end{bmatrix}.$$

3. Write $(1,-1,0)$ in the basis $F = \left\{ \begin{bmatrix}3\\5\\2\end{bmatrix}, \begin{bmatrix}7\\1\\4\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$.

   Solution: The transition matrix from F to the standard matrix is

   $$A = \begin{bmatrix}3 & 7 & 1\\5 & 1 & 1\\2 & 4 & 1\end{bmatrix}$$

   so the transition matrix from the standard basis to F is

   $$A^{-1} = \frac{1}{12}\begin{bmatrix}3 & 3 & -6\\3 & -1 & -2\\-18 & -2 & 24\end{bmatrix}$$

   and

   $$[(1,-1,0)]_F = \frac{1}{12}\begin{bmatrix}3 & 3 & -6\\3 & -1 & -2\\-18 & -2 & 24\end{bmatrix} \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\1/3\\-4/3\end{bmatrix}.$$

4. Write $(2,3,4)$ in the basis $F = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

   Solution:

   The transition matrix from F to the standard basis is

   $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

   so the transition matrix from the standard basis to F is

   $$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

   and

   $$[(2,3,4)]_F = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}.$$

Find the matrix of the operator with respect to the given basis

1. Give the matrix of $L(x,y,z) = (3x+y+z, 5x-2y+z, y+z)$ with respect to $F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

   Solution:

   The matrix of L with respect to the standard basis is

   $$A = \begin{bmatrix} 3 & 1 & 1 \\ 5 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

   The transition matrix from F to the standard basis is

   $$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

   with inverse

   $$S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}.$$

   Thus the matrix of L with respect to F is

   $$S^{-1}AS = \begin{bmatrix} 3 & 6 & 4 \\ 1 & -1/2 & -3/2 \\ 0 & -3/2 & -1/2 \end{bmatrix}.$$

2. Give the matrix of $L(x,y,z) = (2x+3y-z, 4x-y+3z, 2x+z)$ with respect to $F = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

   Solution:

The matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

with

$$S^{-1} = \begin{bmatrix} -3/2 & 1 & -1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}.$$

Thus the matrix of L with respect to F is

$$S^{-1}AS = \begin{bmatrix} -11/2 & -7 & -19/2 \\ 3/2 & 5 & 7/2 \\ 3/2 & 2 & 5/2 \end{bmatrix}.$$

3. Give the matrix of $L(x, y, z) = (-x+4y+2z, 3x-5y+2, 3x+2y)$ with respect to $F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution: The matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -5 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with inverse

$$S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Thus the matrix of L with respect to F is

$$S^{-1}AS = \begin{bmatrix} 0 & 5 & 3 \\ 5 & -2 & 0 \\ 0 & -2 & -4 \end{bmatrix}.$$

4. Give the matrix of $L(x, y, z) = (2x-y, 3x+y+4z, x+2y+z)$ with respect to $F = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution:

The matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$S = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with inverse

$$S^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Thus the matrix of L with respect to F is

$$S^{-1}AS = \begin{bmatrix} 7 & 4 & 2 \\ 1 & 0 & -1 \\ -18 & -11 & 3 \end{bmatrix}.$$

Angles and Magnitudes

1. Compute

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 1 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ 7 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: $20, 44, -1, -15.$

2. Find the magnitudes and corresponding unit vectors for

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ -1 \\ -3 \end{bmatrix}.$$

Solution: $\sqrt{9+1+4} = \sqrt{13}, \sqrt{25+144} = 13, \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}, \sqrt{49+1+9} = \sqrt{59}.$

3. Find $\text{proj}_{\mathbf{v}} \mathbf{u}$ for

(a) $\mathbf{u} = (5, 2), \mathbf{v} = (-3, 4)$
(b) $\mathbf{u} = (2, 1), \mathbf{v} = (7, 1)$
(c) $\mathbf{u} = (3, 1, 4), \mathbf{v} = (2, 1, 1)$
(d) $\mathbf{u} = (2, 1, 1), \mathbf{v} = (-4, -1, -1)$
(e) $\mathbf{u} = (5, 0, 0), \mathbf{v} = (3, 2, 1).$

Solution:

(a) $\frac{-7}{25} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

(b) $\frac{15}{50} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

(c) $\frac{11}{6}\begin{bmatrix}2\\1\\1\end{bmatrix}$

(d) $\frac{-10}{18}\begin{bmatrix}-4\\-1\\-1\end{bmatrix}$

(e) $\frac{15}{14}\begin{bmatrix}3\\2\\1\end{bmatrix}$

Find parametric and normal equations for

1. $y = 5x + 2$

   Solution:

   $$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} + t\begin{bmatrix}1\\5\end{bmatrix}$$

   or

   $$\begin{bmatrix}-5 & 1\end{bmatrix}\begin{bmatrix}x-0\\y-2\end{bmatrix} = 0.$$

2. $2y = 3x + 4$

   Solution:

   $$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} + t\begin{bmatrix}2\\3\end{bmatrix}$$

   or

   $$\begin{bmatrix}3 & -2\end{bmatrix}\begin{bmatrix}x-0\\y-2\end{bmatrix} = 0.$$

3. $y = 3x + 1, z = -2x + 3$

   Solution: For the parametric form we need a point and a slope. So we have

   $$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\1\\3\end{bmatrix} + t\begin{bmatrix}1 & 3 & -2\end{bmatrix}.$$

   For the normal form, we need to find a 2×3 matrix, which we get by putting our equation into standard form. Thus the normal form of the equation is

   $$\begin{bmatrix}3 & 1 & 0\\-2 & 0 & 1\end{bmatrix}\begin{bmatrix}x-0\\y-1\\z-3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

4. The line through $(3, 4)$ and $(1, 7)$

5. The line through $(0, 1)$ and $(6, -3)$

6. $z = 3x + 2y - 2$

   Solution: The parametric form is

   $$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\-2\end{bmatrix} + s\begin{bmatrix}1\\0\\3\end{bmatrix} + t\begin{bmatrix}0\\1\\2\end{bmatrix}$$

To find the normal form, we need a vector normal to both $(1,0,3)$ and $(0,1,2)$. If we set the third coordinate to be 1, we see the first must be -3 and the second -2, so the normal vector is $\mathbf{n} = (-3,-2,1)$, and the equation is

$$\begin{bmatrix} -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x-0 \\ y-0 \\ z+2 \end{bmatrix} = 0.$$

7. $z = -x + 4y + 3$
8. The plane through $(0,2,1), (5,2,1), (6,3,4)$

   Solution: We take $(0,2,1)$ as our base point. To get two vectors in the plane, we compute

$$\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

   so the parametric form is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}.$$

   To find the normal form, we need a vector orthogonal to $(5,0,0)$ and to $(6,2,3)$. We row-reduce a matrix:

$$\begin{bmatrix} 5 & 0 & 0 \\ 6 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

   So we see an element of the kernel is $(0,-3,2)$. Thus the normal form is

$$\begin{bmatrix} 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x-0 \\ y-2 \\ z-1 \end{bmatrix} = 0.$$

9. The plane through $(4,1,1), (-2,3,1), (5,-2,3)$.

Find the nearest point and the distance between

1. The point $(3,1)$ and the line $y = 4x + 2$

   Solution: We subtract 2 from all the y coordinates to make the line go through the origin.

   We want to project the vector $(3,-1)$ onto the vector $(1,4)$. So we compute

$$\begin{aligned} \mathrm{proj}_{(1,4)}(3,-1) &= \frac{(3,-1)\cdot(1,4)}{(1,4)\cdot(1,4)} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \mathrm{proj}_{(1,4)}(3,-1) &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \frac{-1}{17} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 52/17 \\ -21/17 \end{bmatrix} \\ \left\| \begin{bmatrix} 52/17 \\ -21/17 \end{bmatrix} \right\| &= \frac{1}{17}\sqrt{52^2 + 21^2} = \frac{\sqrt{3145}}{17}. \end{aligned}$$

   Thus the point on the line $y = 4x+2$ that is nearest to $(3,1)$ is $(-1/17, -4/17) + (0,2) = (-1/17, 30/17)$, and the distance between the point and the line is $\frac{\sqrt{3145}}{17}$.

2. The point $(2,-2)$ and the line $y = 3x - 7$

   Solution: We add 7 to all y coordinates to make the line go through the origin.

   We want to project $(2,5)$ onto the vector $(1,3)$. We compute

$$\begin{aligned} \operatorname{proj}_{(1,3)} \begin{bmatrix} 2 \\ 5 \end{bmatrix} &= \frac{(2,5)\cdot(1,3)}{(1,3)\cdot(1,3)} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{17}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \operatorname{proj}_{(1,3)} \begin{bmatrix} 2 \\ 5 \end{bmatrix} &= \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \frac{17}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/10 \\ -1/10 \end{bmatrix} \\ \left\| \begin{bmatrix} 3/10 \\ -1/10 \end{bmatrix} \right\| &= \frac{1}{10}\sqrt{3^2+1^2} = \frac{\sqrt{10}}{10}. \end{aligned}$$

   Thus, the point on the line closest to $(2,-2)$ is $(17/10, 51/10) - (0,7) = (17/10, -19/10)$, and the distance between them is $\frac{\sqrt{10}}{10}$.

3. The point $(2,1,4)$ and the line $z = 3x, y = 2x + 2$

   Solution: We subtract 2 off the y coordinates.

   We want to project the vector $(2,-1,4)$ onto the vector $(1,2,3)$. We compute

$$\begin{aligned} \operatorname{proj}_{(1,2,3)}(2,-1,4) &= \frac{(2,-1,4)\cdot(1,2,3)}{(1,2,3)\cdot(1,2,3)} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{12}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - \frac{6}{7} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} &= \begin{bmatrix} 8/7 \\ -19/7 \\ 10/7 \end{bmatrix} \\ \left\| \begin{bmatrix} 8/7 \\ -19/7 \\ 10/7 \end{bmatrix} \right\| &= \frac{1}{7}\sqrt{8^2+19^2+10^2} = \frac{5\sqrt{3}}{\sqrt{7}}. \end{aligned}$$

   Thus the closest point is $(6/7, 12/7, 18/7) + (0,2,0) = (6/7, 26/7, 18/7)$, and the distance between them is $5\sqrt{3}/\sqrt{7}$.

4. The point $(2,4,4)$ and the plane $z = 3x + 2y + 1$

   Solution: We see that the normal vector to the plane is $\mathbf{n} = (3,2,-1)$. We subtract 1 from all the z coordinates to get the plane through the origin, and we want to project $(2,4,3)$ onto $(3,2,-1)$. We compute

$$\begin{aligned} \operatorname{proj}_{\mathbf{n}}(2,4,3) &= \frac{(2,4,3)\cdot(3,2,-1)}{(3,2,-1)\cdot(3,2,-1)} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \frac{11}{14} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \operatorname{proj}_{\mathbf{n}}(2,3,4) &= \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \frac{11}{14} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -5/14 \\ 34/14 \\ 53/14 \end{bmatrix} \\ \left\| \begin{bmatrix} 33/14 \\ 22/14 \\ -11/14 \end{bmatrix} \right\| &= \frac{1}{14}\sqrt{33^2+22^2+11^2} = 11\sqrt{14}. \end{aligned}$$

   Thus the distance between the point and the plane is $11\sqrt{14}$, and the point on the plane closest to the point is $\frac{1}{14}(-5,34,53) + (0,0,1) = \frac{1}{14}(-5,34,67)$.

   (Notice that these problems are different, since we're not projecting onto the plane; we're projecting onto the normal vector instead!)

5. The point $(5, 3, 1)$ and the plane $z = 4x + 2$

   Solution: The normal vector is $\mathbf{n} = (4, 0, -1)$. We subtract 2 off of all the z coordinates to make the plane go through the origin. Then we want to project $(5, 3, -1)$ onto $(4, 0, -1)$. We compute

$$\text{proj}_{\mathbf{n}} \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \frac{(5,3,-1)\cdot(4,0,-1)}{(4,0,-1)\cdot(4,0,-1)} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \frac{21}{17} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} - \text{proj}_{\mathbf{n}} \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} - \frac{21}{17} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/17 \\ 3 \\ 4/17 \end{bmatrix}$$

$$\left\| \frac{21}{17} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \right\| = \frac{21}{17}\sqrt{4^2+1} = \frac{21\sqrt{17}}{17}.$$

   Thus the distance between the point and the plane is $\frac{21\sqrt{17}}{17}$, and the point on the plane nearest the point is $(1/17, 3, 4/17) + (0, 0, 2) = (1/17, 3, 38/17)$.

6. The point $(2, 2, 2)$ and the plane through $(1, 1, 1), (3, 4, 5), (7, 2, 1)$.

   Solution: We see that two vectors in the plane are $(2, 3, 4)$ and $(6, 1, 0)$. We want a vector which is perpendicular to both of these, so we compute

$$\begin{bmatrix} 2 & 3 & 4 \\ 6 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 4 \\ 0 & -8 & -12 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1/2 \\ 0 & 2 & 3 \end{bmatrix}$$

   so we see an element of the kernel is $(1, -6, 4)$. So we can take this to be the normal vector $\mathbf{n}$. We subtract 1 from every coordinate, and want to project $(1, 1, 1)$ onto $(1, -6, 4)$. We compute

$$\text{proj}_{\mathbf{n}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{(1,1,1)\cdot(1,-6,4)}{(1,-6,4)\cdot(1,-6,4)} \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix} = \frac{-1}{53} \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \text{proj}_{\mathbf{n}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1}{53} \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 54/53 \\ 47/53 \\ 57/53 \end{bmatrix}$$

$$\left\| \frac{-1}{53} \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix} \right\| = \frac{1}{53}\sqrt{1+6^2+4^2} = \frac{1}{53}\sqrt{53} = \frac{\sqrt{53}}{53}.$$

   Thus thedistance between the point and the plane is $\frac{\sqrt{53}}{53}$, and the point on the plane closest to $(2, 2, 2)$ is $\frac{1}{53}(54, 47, 57) + (1, 1, 1) = \frac{1}{53}(107, 100, 110)$.