Math 214 Final Exam
Practice Problems

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

I will post at least some solutions soon. I probably won't get a full solutions document written up, though we'll see what happens.

Proofs

1. If λ is an eigenvalue of A then prove that λ^{-1} is an eigenvalue of A^{-1}.
2. Suppose $S, T : V \to V$ are linear and have the property that $S(T(\mathbf{v})) = T(S(\mathbf{v}))$ for every $\mathbf{v} \in V$. If $\mathbf{v}$ is an eigenvector of T, prove that $S(\mathbf{v})$ is also an eigenvector of T.
3. Suppose $L : V \to V$ is a linear transformation of rank k. Prove that L has at most $k+1$ distinct eigenvalues.

Things to Ponder

1. Find a 4×4 matrix with no real eigenvalues. Is it possible to find a 3×3 matrix with no real eigenvalues?
2. In class I said that $\mathrm{Tr}(A)\,\mathrm{Tr}(B) = \mathrm{Tr}(AB)$. This was an error. Find a counterexample.

   Find a matrix A such that $\mathrm{Tr}(A^2) < 0$.
3. What happens if you use the Gram-Schmidt process on a set of vectors that isn't linearly independent?

Diagonalization Theory

1. In class we saw that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

   Multiply out the three matrices on the right and confirm that this works.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What are the eigenvalues of A? Is $A^2 = A$? Why not?

3. Show the following pairs of matrices are not similar:

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 8 & -2 \\ 0 & 0 & 10 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 5 & 3 & 12 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Find the characteristic polynomials, eigenvalues (with algebraic multiplicity), and bases for the eigenspaces, of the following matrices.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Determinants

1. Find all values of k for which $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$ is invertible.

2. Compute the determinants of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$$

Diagonalization

For each of the following matrices, determine whether it is diagonal. If it is, diagonalize it, then computeA^5.

1. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Orthogonality and Projection

1. Suppose $\|\mathbf{u}\| = 3, \|\mathbf{u}+\mathbf{v}\| = 4, \|\mathbf{u}-\mathbf{v}\| = 6$. Find $\|\mathbf{v}\|$.

2. Find the orthogonal complement (in $\mathbb{R}^n$) of the following spaces:

$$W = \{2x - y = 0\}$$
$$W = \{2x - y + 3z = 0\}$$
$$W = \{(t, -t, 3t)\}$$
$$W \,\text{span}\{(1,-1,3,-2),(0,1,-2,1)\}.$$

3. Find the orthogonal decomposition of
   (a) $(7,-4)$ with respect to $\text{span}\{(1,1)\}$
   (b) $(1,2,3)$ with respect to $\text{span}\{(2,-2,1),(-1,1,4)\}$
   (c) $(4,-2,3)$ with respect to $\text{span}\{(1,2,1),(1,-1,1)\}$
   (d) $(3,2,-3,4)$ with respect to $\text{span}\{(2,1,0,1),(0,-1,1,1)\}$.

4. Find the distance between, and nearest point on,
   (a) $(2,2)$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   (b) $(0,1,0)$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$
   (c) $(2,2,2)$ and $x + y - z = 0$
   (d) $(0,0,0)$ and $x - 2y + 2z = 1$.