Math 214 Final Exam
Practice Problem Solutions

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

These are partial solutions. I should be able to finish the solutions tomorrow.

There are almost certainly a couple typos in here. If you think you've found a typo: first download the newest version from the website; I might have found and fixed it already. If not, send me an email or come see me.

Proofs

1. If λ is an eigenvalue of A then prove that λ^{-1} is an eigenvalue of A^{-1}.

   Solution: Let $\mathbf{v} \in E_\lambda$ be an eigenvector with eigenvalue λ. Then $A\mathbf{v} = \lambda\mathbf{v}$, which implies that $A^{-1}(\lambda\mathbf{v})A^{-1}A\mathbf{v} = \mathbf{v}$. Dividing both sides by λ, we have $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

2. Suppose $S, T : V \to V$ are linear and have the property that $S(T(\mathbf{v})) = T(S(\mathbf{v}))$ for every $\mathbf{v} \in V$. If $\mathbf{v}$ is an eigenvector of T, prove that $S(\mathbf{v})$ is also an eigenvector of T.

   Solution: $T(S(\mathbf{v})) = S(T(\mathbf{v})) = S(\lambda\mathbf{v}) = \lambda S(\mathbf{v})$ so $S(\mathbf{v})$ is an eigenvector of T by definition.

3. Suppose $L : V \to V$ is a linear transformation of rank k. Prove that L has at most $k+1$ distinct eigenvalues.

   Solution: Suppose that L has $k+2$ distinct eigenvalues; then it has at least $k+1$ distinct non-zero eigenvalues, which we will denote $\lambda_1, \dots, \lambda_{k+1}$. Let $\mathbf{v}_1, \dots, \mathbf{v}_{k+1}$ be corresponding eigenvectors.

   Since these vectors have distinct eigenvectors, they are linearly independent. Further, each of them is in the image of L, since $L(\mathbf{v}_i/\lambda_i) = \mathbf{v}_i$. Thus the image of L contains $k+1$ linearly independent vectors, and so has dimension at least $k+1$, which contradicts the assumption that the rank of L is k.

Things to Ponder

1. Find a 4×4 matrix with no real eigenvalues. Is it possible to find a 3×3 matrix with no real eigenvalues?

   Solution: We want to find a matrix whose characteristic polynomial has no real roots. The simplest and most obvious such polynomial is $(x^2+1)^2$, so we want to build one of these. The simplest way to do *that* is to find a 2×2 matrix with characteristic polynomial x^2+1 and repeat it twice.

   We've actaully seen this matrix before; it's the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which has characteristic polynomial

   $$\chi(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = (-\lambda)^2 - (-1 \cdot 1) = \lambda^2 + 1.$$

   To get a 4×4 matrix we can glue two copies of this together. We set

   $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

which you can see has characteristic polynomial $\chi_A(\lambda) = (\lambda^2+1)(\lambda^2+1)$. This has no real roots, so the matrix has no real eigenvalues.

For a 3×3 matrix, we would be looking for a degree 3 polynomial with no real roots. No such polynomial exists, so every 3×3 matrix has a real eigenvalue.

(For similar reasons, it is a theorem that every matrix has a *complex* eigenvalue).

2. In class I said that $\operatorname{Tr}(A)\operatorname{Tr}(B) = \operatorname{Tr}(AB)$. This was an error. Find a counterexample.

   Find a matrix A such that $\operatorname{Tr}(A^2) < 0$.

   Solution:

   Solving the second will also solve the first.

   Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\operatorname{Tr}(A) = 0$. But

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \operatorname{Tr}(A^2) = -2 \neq 0^2.$$

   Secretly what's going on here is that A^2 has the eigenvalues $\pm i$, so A^2 has the eigenvalues $(\pm i)^2$, both of which are -1.

3. What happens if you use the Gram-Schmidt process on a set of vectors that isn't linearly independent?

   Solution: When you get to the vector that is a linear combination of the previous vectors, it will equal the sum of its projections onto them. So one of your vectors will be transformed into zero.

   That is, if $\mathbf{e}_3 \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, then $\mathbf{f}_3 = \mathbf{e}_3 - \operatorname{proj}_{\mathbf{e}_1} \mathbf{e}_3 - \operatorname{proj}_{\mathbf{e}_2} \mathbf{e}_3 = 0$.

   Thus the Gram-Schmidt process can be used to turn a spanning set into a basis, by throwing out the vectors that become zero.

Diagonalization Theory

1. In class we saw that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

   Multiply out the three matrices on the right and confirm that this works.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What are the eigenvalues of A? Is $A^2 = A$? Why not?

   Solution: $\chi_A(\lambda) = (1-\lambda)^2$ has roots $1, 1$, so the eigenvalues are 1. We compute that

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq A.$$

   In class we argued that if a diagonalizable matrix has eigenvalues all equal to 1 and 0, then $A^n = A$. This matrix has all eigenvalues 1, but it is not in fact diagonalizable since $\dim E_1 = 1$. Thus the same principle does not hold.

3. Show the following pairs of matrices are not similar:

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 8 & -2 \\ 0 & 0 & 10 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 5 & 3 & 12 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: $\text{Tr}(A) = 5$ and $\text{Tr}(B) = 2$ so the matrices aren't similar.

$\text{Tr}(C) = \text{Tr}(D) = 8$, but $\det(C) = 16$ and $\det(D) = 18$ so the matrices aren't similar.

$\text{Tr}(E) = \text{Tr}(F) = 21$ and $\det(E) = \det(F) = 240$. But the eigenvalues of E are $3, 8, 10$ and the eigenvalues of F are $4, 5, 12$, so the matrices are not similar.

G and H have the same sets of eigenvalues. But G is the identity and so is only similar to itself.

Eigenvalues and Eigenvectors

Find the characteristic polynomials, eigenvalues (with algebraic multiplicity), and bases for the eigenspaces, of the following matrices.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

   Solution: $\chi(\lambda) = -\lambda^3 + 2\lambda^2 - \lambda - 2$ has roots $2, 1, -1$. The corresponding eigenvectors are $(1, 1, 1), (-1, -1, 2), (-1, 1, 0)$.

2. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

   Solution: $\chi(\lambda) = \lambda^2 - \lambda^3$ has roots $1, 0, 0$ with corresponding eigenvectors $(1, 0, 1)$ and $(2, -1, 1)$.

3. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

   Solution: $\chi(\lambda) = -\lambda^3 + 4\lambda^2 - 4\lambda$ has roots $2, 2, 0$, with corresponding eigenvectors $(-1, 0, 1), (-1, 1, 0), (1, 0, 1)$.

4. $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

   Solution: $\chi(\lambda) = -\lambda^3 + 9\lambda^2 - 27\lambda + 27$ has roots $3, 3, 3$, with corresponding eigenvectors $(-1, 0, 1)$ and $(0, 1, 0)$.

5. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

   Solution: $\chi(\lambda) = \lambda^4 - 9\lambda^3 + 18\lambda^2 + 32\lambda - 96$ has roots $4, 4, 3, -2$ with corresponding eigenvectors $(0, 1, 0, 0), (1, 0, 0, 0), (-1, -2, 1, 1), (2, -1, -12, 18)$.

6. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution: $\chi(\lambda) = \lambda^4 - 8\lambda^3 + 23\lambda^2 - 28\lambda + 12$ has roots $3, 1, 2, 2$ with corresponding eigenvalues $(1, 2, 0, 0), (0, 1, 0, 0), (0, 0, -1, 0)$. Note that $\dim E_2 = 1$.

Determinants

1. Find all values of k for which $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$ is invertible.

   Solution: We compute $\det(A) = k(k-2)^2$. We know the matrix is invertible if and only if $\det A \neq 0$, so A is invertible unless $k = 0$ or $k = 2$.

2. Compute the determinants of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$$

   Solution:

$$\begin{array}{ccc} 0 & -2 & -12 \\ 4 & 8 & 0. \end{array}$$

   In particular notice that in the fourth and fifth matrices, we can pick the third row and second column rspectively to make our job much easier; in the sixth matrix, the third row is the sum of the first two so the matrix must be linearly dependent.

Diagonalization

For each of the following matrices, determine whether it is diagonal. If it is, diagonalize it, then computeA^5.

1. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

   Solution: A has eigenvalues $7, 3$ with eigenvectors $(1, 1), (-1, 1)$. This gives us

$$\begin{aligned} U &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ U^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ D = U^{-1}AU &= \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \\ A^5 = UD^5U^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}^5 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 16807 & 0 \\ 0 & 243 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 8525 & 8282 \\ 8282 & 8525 \end{bmatrix}. \end{aligned}$$

2. $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$

   Solution: The eigenvalues are $2, -1$ with corresponding eigenvectors $(1, 1), (2, 1)$. We have

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$D = U^{-1}AU = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^5 = UD^5U^{-1} = U \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} U^{-1} = \begin{bmatrix} -34 & 66 \\ -33 & 65 \end{bmatrix}.$$

3. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

   Solution: The only eigenvalue is 3, and the corresponding eigenvector is $(1, 0, 0)$. Thus the eigenvectors do not span $\mathbb{R}^3$ and so the matrix is not diagonalizable.

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

   Solution: The eigenvalues are $2, -1, 1$ with corresponding eigenvectors $(1, 1, 1), (-1, -1, 2), (-1, 1, 0)$. We compute

$$U = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$U^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & 2 \\ -3 & 3 & 0 \end{bmatrix}$$

$$D = U^{-1}AU = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 = UD^5U^{-1} = U \begin{bmatrix} 32 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1} = \begin{bmatrix} 11 & 10 & 11\ 10 & 11 & 11 \\ 11 & 11 & 10 \end{bmatrix}.$$

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

   Solution: The eigenvalues are $2, 1, 1$ with corresponding eigenvectors $(0, 1, 0)$ and $(0, -1, 1)$. The eigenvectors don't span, so the matrix is not diagonalizable.

6. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

   Solution:

A has eigenvalues $3, 1, 1$ with corresponding eigenvectors $(1, 1, 1), (-1, 0, 1), (0, 1, 0)$. Then we have

$$U = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$U^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$D = U^{-1}AU = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 = UD^5U^{-1} = U \begin{bmatrix} 243 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1} = \begin{bmatrix} 122 & 0 & 121 \\ 121 & 1 & 121 \\ 121 & 0 & 122 \end{bmatrix}.$$

Orthogonality and Projection

1. Suppose $\|\mathbf{u}\| = 3, \|\mathbf{u} + \mathbf{v}\| = 4, \|\mathbf{u} - \mathbf{v}\| = 6$. Find $\|\mathbf{v}\|$.

   Solution: We have

$$\begin{aligned} 9 &= \langle \mathbf{u}, \mathbf{u} \rangle \\ 16 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ 36 &= \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ 52 &= 2\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{v}, \mathbf{v} \rangle = 2 \cdot 9 + 2\langle \mathbf{v}, \mathbf{v} \rangle \\ 34 &= 2\langle \mathbf{v}, \mathbf{v} \rangle \\ \sqrt{17} &= \|\mathbf{v}\|. \end{aligned}$$

2. Find the orthogonal complement (in $\mathbb{R}^n$) of the following spaces:

$$\begin{aligned} &W = \{2x - y = 0\} \\ &W = \{2x - y + 3z = 0\} \\ &W = \{(t, -t, 3t)\} \\ &W \operatorname{span}\{(1, -1, 3, -2), (0, 1, -2, 1)\}. \end{aligned}$$

   Solution:

$$\begin{aligned} W^\perp &= \operatorname{span}\{(1, 2)\} \\ W^\perp &= \operatorname{span}\{(1, 2, 0), (3, 0, -2)\} \\ W^\perp &= \operatorname{span}\{(1, 1, 0), (-3, 0, 1) \\ \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \\ W^\perp &= \operatorname{span}\{(-1, 2, 1, 0), (1, -1, 0, 1)\}. \end{aligned}$$

3. Find the orthogonal decomposition of

   (a) $(7, -4)$ with respect to $\operatorname{span}\{(1, 1)\}$

   Solution:

$$\begin{aligned} \begin{bmatrix} 7 \\ -4 \end{bmatrix}_U &= \operatorname{proj}_{(1,1)} \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \frac{(7, -4) \cdot (1, 1)}{(1, 1) \cdot (1, 1)} \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} \\ \begin{bmatrix} 7 \\ -4 \end{bmatrix}_{U^\perp} &= \begin{bmatrix} 7 \\ -4 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -11/2 \end{bmatrix}. \end{aligned}$$

(b) $(1,2,3)$ with respect to $\text{span}\{(2,-2,1),(-1,1,4)\}$

Solution: The basis we have is orthogonal, so we can just project onto it.

$$\text{proj}_{(2,-2,1)} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{(1,2,3)\cdot(2,-2,1)}{(2,-2,1)\cdot(2,-2,1)} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/9 \\ -2/9 \\ 1/9 \end{bmatrix}$$

$$\text{proj}_{(-1,1,4)} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{(1,2,3)\cdot(-1,1,4)}{(-1,1,4)\cdot(-1,1,4)} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \frac{13}{18} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -13/18 \\ 13/18 \\ 26/9 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_U = \begin{bmatrix} 2/9 \\ -2/9 \\ 1/9 \end{bmatrix} + \begin{bmatrix} -13/18 \ 13/18 \\ 22/9 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix}.$$

(c) $(4,-2,3)$ with respect to $\text{span}\{(1,2,1),(1,-1,1)\}$

Solution:

$$\text{proj}_{(1,2,1)} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \frac{3}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$$

$$\text{proj}_{(1,-1,1)} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \frac{9}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}_U = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -2 \\ 7/2 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 7/2 \\ -2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}.$$

(d) $(3,2,-3,4)$ with respect to $\text{span}\{(2,1,0,1),(0,-1,1,1)\}$.

Solution:

$$\text{proj}_{(2,1,0,1)} \begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix} = \frac{12}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{proj}_{(0,-1,1,1)} \begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix}_U = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ -1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 7/3 \\ -1/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1/3 \\ -8/3 \\ 7/3 \end{bmatrix}.$$

4. Find the distance between, and nearest point on,

(a) $(2,2)$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution: We subtract $(-1,2)$ and so want to find the distance between $(3,0)$ and the line $(x,y) = t(1,-1)$. We project

$$\mathrm{proj}_{(1,-1)} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}$$

so the closest point on the line is $(3/2,-3/2) + (-1,2) = (1/2,1/2)$. The distance is

$$\left\| \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} \right\| = \sqrt{9/2} = \frac{3}{\sqrt{2}}.$$

(b) $(0,1,0)$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

Solution:

We want to project $(-1,0,-1)$ onto $(-2,0,3)$. We have

$$\mathrm{proj}_{(-2,0,3)} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \frac{-1}{13} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix}.$$

So the closest point is $(2/13,0,-3/13) + (1,1,1) = (15/13,1,10/13)$. The distance is

$$\left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 15/13 \\ 1 \\ 10/13 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -15/13 \\ 0 \\ 10/13 \end{bmatrix} \right\| = \frac{1}{13}\sqrt{15^2 + 10^2} = \frac{\sqrt{325}}{13} = \frac{5\sqrt{13}}{13}.$$

(c) $(2,2,2)$ and $x + y - z = 0$

(d) $(0,0,0)$ and $x - 2y + 2z = 1$.