Math 214 Test 1 Solutions

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Problem 1 (4 points each)**.** Are the following statements true or false? Give a short (one sentence or less) explanation.

1. Every vector space contains at least one vector.

   Solution: True, because every vector space contains the zero vector.

2. $\{(a, b, c) : a + b = c + 1\}$ is a subspace of $\mathbb{R}^3$.

   Solution: No, because the zero vector is not an element.

3. Every basis for $\mathcal{P}_n(x)$ has exactly n elements.

   Solution: False, because the dimension is $n + 1$.

4. Every set with exactly one element is linearly independent.

   Solution: False, since $\{\mathbf{0}\}$ is not linearly independent.

5. Every finite-dimensional vector space has a finite basis.

   Solution: True, because this is the definition of finite-dimensional.

Problem 2 (10 points each)**.**

1. Prove that $S = \{(a, b, c) : 3a + 2b + c = 0\}$ is a subspace of $\mathbb{R}^3$.

   Solution: We need to check three things.

   (a) $3 \cdot 0 + 2 \cdot 0 + 0 = 0$ so $\mathbf{0}$ is in the set.

   (b) If $(a, b, c), (d, e, f)$ are elements, then $3a + 2b + c = 0$ and $3d + 2e + f = 0$, so $3(a + d) + 2(b + e) + (c + f) = 0$ and thus $(a, b, c) + (d, e, f)$ is an element.

   (c) If (a, b, c) is an element, then $3a + 2b + c = 0$, so $3(ra) + 2(rb) + rc = 0$, so $r(a, b, c)$ is an element.

   Thus by the subspace theorem, this is a subspace.

2. Prove that $T = \{a_0 + a_2x^2 + a_4x^4\}$ is a subspace of $\mathcal{P}_4(x)$.

   Solution: We need to check three things.

   (a) $0 = 0 + 0x^2 + 0x^4$ is an element of the set.

   (b) If $a_0 + a_2x^2 + a_4x^4$ and $b_0 + b_2x^2 + b_4x^4$ are elements, then $a_0 + a_2x^2 + a_4x^4 + b_0 + b_2x^2 + b_4x^4 = (a_0 + b_0) + (a_2 + b_2)x^2 + (a_4 + b_4)x^4$ is an element.

   (c) If $a_0 + a_2x^2 + a_4x^4$ is an element and r is a scalar, then $r(a_0 + a_2x^2 + a_4x^4) = ra_0 + ra_2x^2 + ra_4x^4$ is an element.

   Thus by the subspace theorem this is a subspace.

Problem 3 (5 points each)**.**

1. Is $S = \{(1, 3), (-2, -6)\}$ a basis for $\mathbb{R}^2$?

   Solution: S is not linearly independent, since $(-2)(1, 3) = (-2, 6)$ so one element is a scalar multiple of the other. Thus it is not a basis.

2. Is $T = \{(1,3),(2,3)\}$ a basis for $\mathbb{R}^2$?

   Solution: Yes. We check linear independence: suppose

$$\begin{bmatrix}0\\0\end{bmatrix} = a\begin{bmatrix}1\\3\end{bmatrix} + b\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}a+2b\\3a+3b\end{bmatrix}.$$

   Then we have $a+2b=0$ and $3a+3b=0$. Then we see that $a=-2b$, so $-6b+3b=0$, so $b=0$ and then $a=0$. Thus the set is linearly independent. Since it is a two-element linearly independent set of $\mathbb{R}^2$, and we know that $\mathbb{R}^2$ is dimension 2, then it must be a basis.

3. Is $U = \{(1,2,1),(1,3,1),(1,4,1)\}$ a basis for $\mathbb{R}^3$?

   Solution: No, because $\frac{1}{2}(1,2,1)+\frac{1}{2}(1,4,1)=(1,3,1)$ so one element is a linear combination of the others. Thus the set is not linearly independent, and thus not a basis.

4. Is $V = \{(1,1,0),(1,1,1),(3,1,1)\}$ a basis for $\mathbb{R}^3$?

   Solution: Yes. We see that it spans, by showing that the standard basis is in the span:

$$\begin{aligned}\mathbf{e}_1 &= \frac{1}{2}(3,1,1) - \frac{1}{2}(1,1,1)\\ \mathbf{e}_3 &= (1,1,1)-(1,1,0)\\ \mathbf{e}_2 &= (1,1,0)-\mathbf{e}_1 = (1,1,0)+\frac{1}{2}(1,1,1)-\frac{1}{2}(3,1,1).\end{aligned}$$

   Thus V spans $\mathbb{R}^3$. Since it is a three-element set which spans $\mathbb{R}^3$, and we know that $\mathbb{R}^3$ is dimension 3, it must be a basis.

Problem 4 (5 points each)**.**

1. Find a basis for $\mathbb{R}^3$ containing $S = \{(1,1,2),(2,2,2)\}$. (Assume that S is linearly independent).

   Solution: We just need a vector that isn't in the span of this set. We guess $(1,0,0)$, and we set

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\1\\2\end{bmatrix} + b\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}a+2b\\a+2b\\2a+2b\end{bmatrix}.$$

   Then we have $a+2b=1, a+2b=0, 2a+2b=0$. The first two equations can't be true simultaneously, so this has no solution. Thus $(1,0,0)$ is not in the span.

   Thus by basis padding, it must be the case that $\{(1,1,2),(2,2,2),(1,0,0)\}$ is a basis for $\mathbb{R}^3$, since it has the correct number of elements.

2. Find a basis for $\mathbb{R}^3$ that is a subset of $T = \{(1,3,3),(2,2,2),(-1,1,1),(0,1,0)\}$. (Assume that T is a spanning set).

   Solution: We need to write one element as a linear combination of the others. We notice that $(1,3,3) = (2,2,2) + (-1,1,1)$ and thus we can write $(1,3,3)$ as a linear combination of the other vectors. By basis reduction we can remove it, and $\{(2,2,2),(-1,1,1),(0,1,0)\}$ is still a spanning set. Since it has the correct number of elements, it must be a basis.

Problem 5 (10 points each)**.**

1. If $\{\mathbf{u},\mathbf{v}\}$ is linearly independent, prove that $\{\mathbf{u}+\mathbf{v},\mathbf{u}-\mathbf{v}\}$ is linearly independent.

   Solution: Suppose

$$\mathbf{0} = a(\mathbf{u}+\mathbf{v}) + b(\mathbf{u}-\mathbf{v}) = (a+b)\mathbf{u} + (a-b)\mathbf{v}.$$

   Then since $\{\mathbf{u},\mathbf{v}\}$ is linearly independent, we know that $a+b=0$ and $a-b=0$. Solving this gives us $a=b$, and thus $2a=0$ so $a=0$, and then $b=0$. So whenever a linear combination of $\{\mathbf{u}+\mathbf{v},\mathbf{u}-\mathbf{v}\}$ is equal to zero, the coefficients must be zero; thus the set is linearly independent by definition.

2. Suppose U, W are subspaces of a vector space V. We define $U \cap W = \{\mathbf{v} : \mathbf{v} \in U \quad \text{and} \quad \mathbf{v} \in W\}$ to be the set of all vectors that are in both U and W, which we call the "intersection" of U and W.

   Prove that $U \cap W$ is a subspace of V.

   Solution: We need to check three things.

   (a) By definition of subspace, we know that $\mathbf{0} \in U$ and $\mathbf{0} \in W$. Thus $\mathbf{0} \in U \cap W$.

   (b) Suppose $\mathbf{v}_1, \mathbf{v}_2 \in U \cap W$. Then $\mathbf{v}_1, \mathbf{v}_2 \in U$, so $\mathbf{v}_1 + \mathbf{v}_2 \in U$ by additive closure. Similarly, $\mathbf{v}_1 \in W, \mathbf{v}_2 \in W$, so $\mathbf{v}_1 + \mathbf{v}_2 \in W$ by additive closure. Thus $\mathbf{v}_1 + \mathbf{v}_2 \in U \cap W$ by definition.

   (c) Suppose $\mathbf{v} \in U \cap W$ and $r \in \mathbb{R}$. Then $\mathbf{v} \in U$ so $r\mathbf{v} \in U$ by scalar multiplicative closure; and $\mathbf{v} \in W$ so $r\mathbf{v} \in W$ by scalar multiplicative closure. Thus $r\mathbf{v} \in U \cap W$ by definition.

   Thus by the subspace theorem, $U \cap W$ is a subspace of V.