# Math 214 Test 3 Solutions

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**Problem 1.** Let $L(x,y,z) = \begin{bmatrix} 3x-2y \\ 2y-z \\ x+y+z \end{bmatrix}$. Compute $L^{-1}$.

**Solution:** The matrix of $L$ is

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We compute

$$\left[\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -5 & -3 & 1 & 0 & -3 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & -5 & -3 & 1 & 0 & -3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & -1/2 & 1 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & -11/2 & 1 & 5/2 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & -1/2 & 1 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -2/11 & -5/11 & 6/11 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/11 & 2/11 & 2/11 \\ 0 & 1 & 0 & -1/11 & 3/11 & 3/11 \\ 0 & 0 & 1 & -2/11 & -5/11 & 6/11 \end{array}\right].$$

Thus

$$L^{-1}(a,b,c) = \frac{1}{11}\begin{bmatrix} 3a+2b+2c \\ -a+3b+3c \\ -2a-5b+6c \end{bmatrix}.$$

**Problem 2.** Let $E = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $F = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Let $\mathbf{u} = 2\mathbf{f}_1 + 3\mathbf{f}_2 + \mathbf{f}_3$. Find $[\mathbf{u}]_E$.

**Solution:** The transition matrix from $F$ to the standard basis is

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

and the transition matrix from $E$ to the standard basis is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We compute

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

so

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Thus

$$B^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

and

$$[\mathbf{u}]_E = B^{-1}A[\mathbf{u}]_F = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ -8 \\ -5 \end{bmatrix}.$$

**Problem 3.** Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -1 & 1 \\ -2 & 5 & 2 \end{bmatrix}$$

with respect to the standard basis. Find the matrix of $L$ with respect to $F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

**Solution:** The transition matrix from $F$ to the standard basis is

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the transition matrix is $S^{-1}$; we compute

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

so

$$S^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Then the matrix we want is

$$B = S^{-1}AS = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & -1 \\ 3 & -1 & 1 \\ -2 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 5 \\ 4 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}.$$

**Problem 4.** (a) Find parametric and normal equations for the plane $U$ through the points $(2,0,0), (4,1,1), (6,5,5)$.

(b) Let $P = (4, 3, 2)$. Find the distance between $P$ and $U$, and find the point on $U$ nearest $P$.
**Solution:**

(a) The parametric equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix}.$$

To find the normal equation, we need the normal vector, so we row reduce

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and an element of the kernel is $(0, 1, -1)$. Thus the normal equation is

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x-2 \\ y-0 \\ z-0 \end{bmatrix} = 0.$$

(b) We begin by subtracting 2 from the x coordinates of everything, and then project $(2, 3, 2)$ onto $(0, 1, -1)$. We get

$$\mathrm{proj}_{(0,1,-1)} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \frac{(2,3,2) \cdot (0,1,-1)}{(0,1,-1) \cdot (0,1,-1)} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} \right\| = \sqrt{0 + 1/4 + 1/4} = \frac{\sqrt{2}}{2}$$

$$\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \mathrm{proj}_{(0,1,-1)} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5/2 \\ 5/2 \end{bmatrix}.$$

Thus the distance is $\sqrt{2}/2$ and the closest point is $(2, 5/2, 5/2) + (2, 0, 0) = (4, 5/2, 5/2)$.

**Problem 5.** (a) Suppose $A, B \in M_n$ are invertible. Prove that $B^{-1}A^{-1}$ is a matrix inverse for $AB$, and thus $AB$ is invertible.

**Solution:** To check that $B^{-1}A^{-1}$ is an inverse for $AB$, we just need to check that their product is the identity. But we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.$$

Thus by definition $B^{-1}A^{-1}$ is an inverse for $AB$, and thus $AB$ is invertible.

(b) Suppose $S : U \to V$ and $T : V \to W$ are both injective. Define $L : U \to W$ by $L(\mathbf{u}) = T(S(\mathbf{u}))$. Prove that $L$ is injective.

**Solution:** Suppose $L(\mathbf{u}) = L(\mathbf{v})$. Then $T(S(\mathbf{u})) = T(S(\mathbf{v}))$. Since $T$ is injective, we know that $S(\mathbf{u}) = S(\mathbf{v})$, and then since $S$ is injective we know that $\mathbf{u} = \mathbf{v}$. Thus whenever $L(\mathbf{u}) = L(\mathbf{v})$, we know that $\mathbf{u} = \mathbf{v}$, and thus by definition $L$ is injective.