Math 214 Test 2 Solutions

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Problem 1. (20 points: 15 and 5)

(a) Solve the system of equations

$$\begin{aligned} x - 4y + 2z &= 2 \\ -x + 3y + z &= 4 \\ 2x - y + z &= 1 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & -4 & 2 & 2 \\ -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -4 & 2 & 2 \\ 0 & -1 & 3 & 6 \\ 0 & 7 & -3 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -10 & -22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 18 & 39 \end{array}\right]$$
$$\to \left[\begin{array}{rrr|r} 1 & 0 & -10 & -22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 1 & 13/6 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & -2/6 \\ 0 & 1 & 0 & 3/6 \\ 0 & 0 & 1 & 13/6 \end{array}\right]$$

so we have $x = -1/3, y = 1/2, z = 13/6$.

(b) Compute

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 8 & 3 \end{bmatrix} =$$

Solution:

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 38 & 8 \end{bmatrix}.$$

Problem 2. (20 points)

Find a bases for the rowspace, columnspace, and nullspace of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 & 2 \\ -1 & -2 & 0 & 0 & 4 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 & 2 \\ -1 & -2 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & -3 & -3 & 0 \\ 0 & 0 & 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 2 & 5 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

Thus the rowspace has basis $\{(1,2,0,0,-4),(0,0,1,0,5),(0,0,0,1,-5)\}$.

The columnspace has basis $\{(1,2,-1),(3,3,0),(2,1,0)\}$.

The nullspace is

$$\{(-2\alpha+4\beta,\alpha,-5\beta,5\beta,\beta)\}=\{\alpha(-2,1,0,0,0)+\beta(4,0,-5,5,1)\}$$

so a basis is $\{(-2,1,0,0,0),(4,0,-5,5,1)\}$.

Problem 3. (15 points)

Find the inverse of

$$\begin{bmatrix}2&1&2\\0&1&1\\1&0&1\end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|ccc}2&1&2&1&0&0\\0&1&1&0&1&0\\1&0&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&1&0&0&1\\2&1&2&1&0&0\\0&1&1&0&1&0\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&0&1&0&-2\\0&1&1&0&1&0\end{array}\right]$$
$$\to\left[\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&0&1&0&-2\\0&0&1&-1&1&2\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&1&-1&-1\\0&1&0&1&0&-2\\0&0&1&-1&1&2\end{array}\right].$$

Thus the inverse is

$$\begin{bmatrix}1&-1&-1\\1&0&-2\\-1&1&2\end{bmatrix}.$$

Problem 4. (15 points)

Let $f(x,y,z)=(x-y,y-z,z-x)$.

(a) Prove that f is linear.

(b) Find the matrix of f with respect to the standard basis.

Solution:

(a) We check:

$$\begin{aligned}f(r(x,y,z))&=(rx-ry,ry-rz,rz-rx)=r(x-y,y-z,z-x)=rf(x,y,z)\\f((x_1,y_1,z_2)+(x_2,y_2,z_2))&=(x_1+x_2-(y_1+y_2),(y_1+y_2)-(z_1+z_2),(z_1+z_2)-(x_1+x_2))\\&=(x_1-y_1,y_1-z_1,z_1-x_1)+(x_2-y_2,y_2-z_2,z_2-x_2)\\&=f(x_1,y_1,z_1)+f(x_2,y_2,z_2).\end{aligned}$$

Thus f is linear by definition.

(b) We have

$$\begin{aligned}f(1,0,0)&=(1,0,-1)\\f(0,1,0)&=(-1,1,0)\\f(0,0,1)&=(0,-1,1)\\A&=\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix}.\end{aligned}$$

Problem 5. (30 points: 15 points each)

(a) Let $L : \mathbb{R}^2 \to \mathcal{P}_2(x)$ be given by $f(a,b) = b + ax + (a+b)x^2$. Let $E = \{(1,1),(1,-1)\}$ and $F = \{1, x, x^2\}$ be bases for $\mathbb{R}^2$ and $\mathcal{P}_2(x)$ respectively. Find a matrix for L with respect to E and F.

Solution:

$$\begin{aligned} L(1,1) &= 1 + x + 2x^2 = \mathbf{f}_1 + \mathbf{f}_2 + 2\mathbf{f}_3 \to (1,1,2) \\ L(1,-1) &= -1 + x = -\mathbf{f}_1 + \mathbf{f}_2 \to (-1,1,0) \\ A &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}. \end{aligned}$$

(b) Let $L : U \to U$ be a linear transformation, and $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Let A be the matrix representing L with respect to E. Suppose that $L(\mathbf{u}) = \mathbf{0}$ for some $\mathbf{u} \neq \mathbf{0}$. Prove that the rank of A is not n.

Solution:

We know there is some $\mathbf{u} \in U$ such that $\mathbf{u} \neq \mathbf{0}$, and $L(\mathbf{u}) = \mathbf{0}$. Then since $\mathbf{u} \neq 0$ we know that $[\mathbf{u}]_E \neq 0$, but since $L(\mathbf{u}) = \mathbf{0}$ we know that $A[\mathbf{u}]_E = [L(\mathbf{u})]_F = [\mathbf{0}]_F = \mathbf{0}$.

Thus $[\mathbf{u}]_E$ is a nonzero element of the nullspace of A, so the nullspace is nontrivial and the nullity is not zero.

Suppose the rank of A is n. Then since A has n columns, by the rank-nullity theorem, the nullity is zero, which is a contradiction.