NAME:___________________________ start time:________ finish:_______

COMPREHENSIVE EXAM IN MATHEMATICS
SPRING 2008

No books, notes, or calculators allowed. You have **three** hours. Remember that the *presentation* of your work will be evaluated, not just your final answers.

problem	score		
1			
2			
3			
4			
5		total calc 1:	out of:
6			
7			
8			
9			
10		total calc 2:	out of:
11			
12			
13			
14			
15		total multi:	out of:
16			
17			
18			
19			
20		total discrete:	out of:
21			
22			
23			
24			
25		total linear:	out of:

CALCULUS I

(1) The initial value problem (IVP) $z' = 3z + 1, \quad z(2) = 9$ defines some function $z(t)$.

(a) Find an equation for a tangent line to $z(t)$, and use it to approximate $z(1.5)$.

(b) Use a second derivative to determine whether your approximation is an overestimate or an underestimate of $z(1.5)$. Explain your answer.

(2) Let $f(x) = \lfloor x \rfloor$ be the "floor function," which equals the greatest integer less than or equal to x.

(a) Sketch the graph of f.

(b) Is the following statement true or false? Explain. "For all x, $f'(x) = 0$, because the tangent line at every point is horizontal."

(3) A function f is *even* if $f(-x) = f(x)$ for all x in its domain, and *odd* if $f(-x) = -f(x)$ for all x in its domain. Suppose that f is an even function (continuously differentiable). Use the chain rule to show that f' is an odd function.

(4) Patrick McCormick is campaigning in Piscataway. Suppose that $P(x) = 17 + \dfrac{50x^2}{x^2 + 75}$ is the percentage of voters who will vote for McCormick, if the McCormick campaign spends x thousands of dollars ($x \geq 0$). Find out how many thousands of dollars should be spent in order to maximize the *rate of change* of percentage of voters supporting McCormick; that is, find the point of diminishing returns. [It will help to simplify as you go along, including canceling, and factoring out constants.]

Note: Choose to do *either* Problem 5, *or* April Fools Problem 5.

(5) The following function comes from a computation involving probability and discrete math: $f(p) = ke^{-pn} + k^2 p$, where k and n are positive real constants.

Use calculus to show that the minimum of this function on the interval $0 \le p \le 1$ occurs at $p = \frac{1}{n}\ln(\frac{n}{k})$.

(April Fools 5) The following function comes from a computation involving probability and discrete math: $f(p) = ke^{-pn} + k^2 p$, where k and n are positive real constants.

(a) Use calculus to show that the minimum of this function on the interval $0 \le p \le 1$ occurs at $p = \frac{1}{n}\ln(\frac{n}{k})$.

(b) Show that if $k = c(\frac{n}{\ln n})^{\frac{1}{2}}$, where c is a positive constant, and $p = \frac{1}{n}\ln(\frac{n}{k})$, then we can rewrite $f(p)$ as $\dfrac{c^2}{\ln n}\left[1 + \frac{1}{2}\ln n - \ln c - \frac{1}{2}\ln(\ln n)\right]$.

(c) Suppose we want the expression in (b) to be strictly less than 1. Show that $\lim_{n\to\infty} \dfrac{c^2}{\ln n}\left[1 + \frac{1}{2}\ln n - \ln c - \frac{1}{2}\ln(\ln n)\right] = \dfrac{c^2}{2}$, and conclude that the expression in (b) will be less than 1 if we choose $c < \sqrt{2}$, and choose n large enough.

(6) Divide the interval $[2,3]$ into n subintervals, each of length Δx, and pick one representative point in each subinterval. Call the representative point in the i^{th} subinterval x_i. Compute the exact value of $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{\Delta x \ln(x_i)}{x_i}\right]$.

(7) Suppose the function f is represented by the power series

$$f(x) = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots + (-1)^k \frac{x^k}{2^k} + \ldots$$

(a) Find the domain of f.

(b) Find $f(0)$ and $f(1)$.

(8) (a) Draw the graph of $\cos(x)$, for $-2\pi \leq x \leq 2\pi$.

(b) Determine the value of $\int_0^{\pi} \cos x \, dx$, by discussing the graph, *not* by doing computations.

(c) Verify your answer to (b) by using the Fundamental Theorem of Calculus.

(9) Suppose g is a differentiable function such that g and g' have the values in the table.

x	-2	0	2	4
$g(x)$	1	2	3	4
$g'(x)$	5	6	7	8

(a) $\int_0^2 x \, g'(x^2) \, dx =$

(b) $\int_0^2 x \, g'(x) \, dx =$

(10) (a) Find the Maclaurin series for $\int_0^x \frac{1}{1+t^3}\,dt$. (You may think of this function as $\int \frac{1}{1+x^3}\,dx$, with constant of integration equal to zero.)

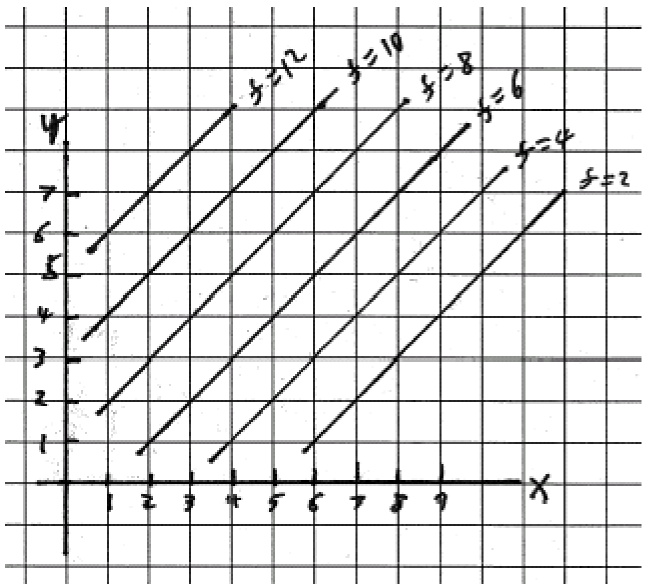
(b) Use the first three (nonzero) terms of your answer to (a) to estimate $\int_0^{\frac{1}{2}} \frac{1}{1+t^3}\,dt$. You need not simplify your answer. (It's possible to do part (b) without doing part (a)! But it's much easier if you can do (a) first.)

MULTIVARIABLE CALCULUS

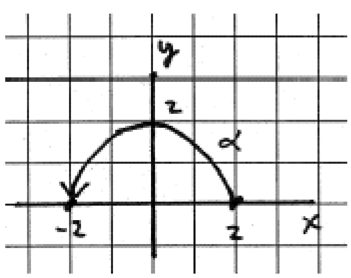
Note: In this section, an arrow above a letter (e.g., $\vec{v}$ or $\vec{F}$) indicates a vector or a vector-valued function.

(11) Find an equation for the plane tangent to the surface $x\sin(yz) + ze^y - x^3yz = 4$ at the point $(2,0,4)$.

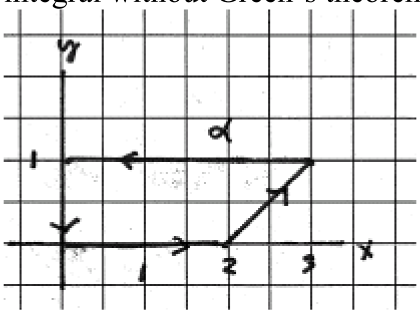
(12) A contour diagram for a function $f(x, y)$ is shown. Let $\vec{v} = (3, -1)$. Use the contour diagram to estimate $f_{\vec{v}}(3,6)$, the directional derivative of f at the point $(3,6)$ in the direction of $\vec{v}$.



(13) Compute the line integral $\int_{\alpha} \vec{F} \cdot d\vec{r}$, where α is the semicircle shown, and $\vec{F}(x, y) = (y, -x)$.



(14) Use Green's theorem to compute the line integral $\int_{\alpha} \vec{F} \cdot d\vec{r}$, where α traverses the perimeter of the quadrilateral shown, and $\vec{F}(x, y) = (e^{y}, xe^{y} + x^{2})$. [Note: It's possible to compute the line integral without Green's theorem for partial credit, but it will take *much* longer that way.]



(15) Suppose your predicted score on a HMM (Hard Math exaM) is given by $f(x, y) = 2x + y^{2}$, where x is the number of hours you spend outlining the material, and y is the number of hours you spend working practice problems. So far, you've spent 3 hours on the outline, and 1 hour working problems. How should you allot your time in the next hour (how many minutes working on the outline, and how many minutes working problems), so that your predicted score on the HMM *increases most quickly*?

DISCRETE MATH

(16) Show that the propositions $p \vee (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

(17) Suppose that $f : A \to B$ and $g : B \to C$ are two given functions. Answer each question, giving a proof or counterexample as justification.

(a) If g is one-to-one and f is onto, then must $g \circ f$ be one-to-one?

(b) If $g \circ f$ is onto and f is one-to-one, then must g be onto?

(18) Let R be a relation on the set of integers defined by $(x, y) \in R$ if and only if $x + y \equiv 0 \pmod 2$.

(a) Show that R is an equivalence relation.

(b) What are the equivalence classes of R?

(19) (a) In how many ways can a committee of four be formed from a pool of nine people?

(b) In how many ways can a committee consisting of a chair, a secretary, and two other members be formed from a pool of nine people?

(c) In how many ways can a committee of four be chosen from a pool of four men and five women, if the committee must contain at least one man and one woman?

(20) Show that $12^n + 2(5^{n-1})$ is divisible by 7 for every positive integer n.

LINEAR SYSTEMS

Note: In this section, an arrow above a letter (e.g., $\vec{v}$) indicates a vector.

(21) (a) Give the definitions of the column space, row space, and null space of a matrix A.

(b) Given a matrix A and a vector $\vec{b}$, if the equation $A\vec{x} = \vec{b}$ has a solution, then is $\vec{b}$ necessarily in $\text{col}(A)$, $\text{row}(A)$, $\text{null}(A)$, or none of the above? Explain your reasoning.

(22) Let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ be four arbitrary vectors in $\Re^3$. Determine whether each of the following is true or false. Just write T or F in front of each, without explanation.

- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is necessarily linearly dependent
- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ necessarily spans $\Re^3$
- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is necessarily a basis for $\Re^3$
- The span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is necessarily a subspace of $\Re^3$
- One of the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ is necessarily a linear combination of the rest
- Each of the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ is necessarily a linear combination of the rest
- Each of the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ is necessarily in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

(23) Suppose A is an invertible $n \times n$ matrix, where $n > 1$. Determine whether each of the following is true or false. Just write T or F in front of each, without explanation.

- The equation $A\vec{x} = 0$ has a nonzero solution
- For every vector $\vec{b}$ in $\Re^n$, the equation $A\vec{x} = \vec{b}$ always has exactly one solution
- The rows of A form a basis for $\Re^n$
- The null space of A has dimension n
- $\text{rref}(A)$ is the identity matrix (reduced row echelon form)
- There exists a nonzero square matrix B such that $AB = 0$
- A^{-1} is an invertible matrix

(24) (a) Give the definition of the projection of a vector $\vec{v}$ onto a vector $\vec{w}$ (this is asking for the "formula" we normally use to compute projections).

(b) Does the definition of projection imply that the projection of $\vec{v}$ onto $\vec{w}$ is necessarily parallel to $\vec{v}$, necessarily parallel to $\vec{w}$, or neither? Explain your reasoning.

(25) (a) Give the definition of "vector $\vec{v}$ is an eigenvector of matrix A".

(b) Prove or disprove: if $\vec{v}$ is an eigenvector of A, then any nonzero scalar multiple of $\vec{v}$ is also an eigenvector of A.