MATHEMATICS COMPREHENSIVE EXAMINATION, SPRING 2006

NAME: ____ Start Time: _____________

Each question is worth 4 points. You have three hours. Show your work!

Single Variable Calculus

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1. Suppose you know that $f(t) = \int_1^t g(s)\,ds$ and that $g(1) = 2$.

   a. Find $f(1)$ and $f'(1)$.

   b. Find the equation for the line tangent to the graph of f at the point $(1, f(1))$.

2. Consider the initial value problem $y' = -y^2 \quad \text{and} \quad y(0) = 2$.

   a. Use Euler's Method with stepsize $\Delta x = 1/2$ to estimate $y(1)$.

   b. Briefly explain how to obtain a better estimate of $y(1)$.

3. The following table of values for the *first* derivative $f'(x)$ is given:

x	0	1	2	3	4
$f'(x)$	4	3	0	-3	-4

a. Estimate the *second* derivative $f''(0)$. Show your work.

b. True or False (explain): The graph of the function f is concave up at $x = 0$.

4. Suppose $f(g(x)) = x$, $g(0) = 1$, and $f'(1) = 2$. Find $g'(0)$.

5. Consider the function $h(x) = |x|$. For each of the following, either evaluate the expression or briefly explain why it doesn't exist:

   a. $h'(0) =$

   b. $\int_{-1}^{1} h(x)\, dx =$

6. $\int x \ln x dx =$

7. A hiker starts hiking at 9 am. There are some hills; she walks faster downhill than uphill. Her *speed* is $s(t) = 1.5 - \sin(\pi t)$ miles per hour. Here t is *hours elapsed since 9 am*.

    a. How *far* did she hike by 1 pm?

    b. How many hills did she climb during that time? Explain.

8. Consider the infinite series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$.

    a. This series converges when $x = 1$. Why?

    b. This series is the Taylor series of a certain function $f(x)$ expanded about the origin. What is the value of $f''(0)$? Do *not* differentiate the series to find out!

9. Find the *area* enclosed *between* the graph of the parabola $f(x) = 2 - x^2$ and the line $g(x) = \frac{1}{2}x$. *Do not bother to simplify your final answer!*

10. Evaluate $\lim_{b\to\infty} \int_0^b e^{-x}\, dx$.

Multivariable/Vector Calculus

11. The temperature in deg F on the surface of a 10 ft. by 8 ft. plate glass window is given by

$$T(x, y) = 72 + y + \frac{1}{80}xy, \qquad 0 \leq x \leq 10,\ 0 \leq y \leq 8.$$

   a. Find the gradient $\nabla T(x, y)$.

   b. If a fly were located at the point $(1, 1)$, in which direction would it start moving if it wanted to increase its temperature the most? Your answer should be a 2-dimensional vector.

12. The function $F(x, y) = x^2 - y^3$ is differentiable at the point $(2, 3)$. Find the equation for the plane tangent to the graph of F at this point.

13. Once again, consider the temperature in deg F on the surface of a 10 ft. by 8 ft. plate glass window given by

$$T(x, y) = 72 + y + \frac{1}{80}xy, \qquad 0 \le x \le 10,\ 0 \le y \le 8.$$

Starting at the point $(10, 5)$, a fly walks across the window along the curve $(x(t), y(t)) = (10 - t, t + 5)$ for $0 \le t \le 1$ minute.

a. At what point on this path was the temperature greatest?

b. What was the value of that greatest temperature?

14. $\displaystyle\int_0^1 \int_z^3 1 + w^2 z \; dw \, dz =$

15. Consider the vector field $\vec{F}(x,y) = (x,y)$ and the curve $\vec{r}(t) = (\cos t, \sin t), 0 \leq t < 2\pi$.

    a. Without calculating it, clearly explain why the line integral $\int_{t=0}^{t=2\pi} \vec{F} \cdot d\vec{r} = 0$.

    b. Then confirm that $\int_{t=0}^{t=2\pi} \vec{F} \cdot d\vec{r} = 0$ by showing the details of the calculation.

Linear Systems

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16. a. Prove or disprove: If λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m.

    b. Find all the eigenvalues of the permuation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

17. This problem concerns subspaces of a vector space $\mathbf{R}^2$.

    a. Prove or disprove: $V = \{(0,1) + c(1,1),\, c \in \mathbf{R}\}$ is a subspace of $\mathbf{R}^2$.

    b. Find a subpace of $\mathbf{R}^2$ *orthogonal* to the subspace $W = \{c(1,1),\, c \in \mathbf{R}\}$.

18. a. Define what it means for a finite set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ to be *linearly independent*.

    b. Is the following set of three vectors in $\mathbf{R}^2$ linearly independent? Explain your answer.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

19. Use the fact that $\det(AB) = \det(A)\det(B)$ to prove:

$$\text{If } A \text{ is nonsingular, then } \det(A^{-1}) = 1/\det(A).$$

20. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 0 & 6 & 10 \end{bmatrix}$.

a. Find a *basis* for the nullspace of A.

b. Either find the general solution of $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, or explain why such a solution doesn't exist.

Discrete Mathematics

Please answer the following questions without working out the permutations, combinations, factorials or powers (i.e. do not simplify your answers).

21. Let $A = \{(p, q) | p, q \in \mathbf{Z},\ q \neq 0\}$. Define the relation $\sim$ on A by

$$(p, q) \sim (r, s) \iff p \cdot s = r \cdot q$$

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Prove or disprove: $\sim$ is an *equivalence* relation on A.

22. Prove or disprove: "The following logical proposition is true no matter what the truth values of propositions p and q are: "

$$(p \to q) \to (\neg q \to \neg p).$$

23. An ice cream shop has 21 flavors of ice cream. You can order either one or two scoops on a cone.

    a. If the order in which the scoops are placed on the cone *doesn't* matter, how many different ways can you order a cone from this ice cream shop?

    b. If the order in which the scoops are placed on the cone *does* matter, how many different ways can you order a cone from this ice cream shop?

24. a. The following statement is false! Correct it.

    "A function $f : X \to Y$ is one-to-one if and only if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$."

    b. How many *onto* functions are there from X to Y if $|X| = 3$ and $|Y| = 2$?

25. Prove that the four-digit positive integer $n = abcd$ is divisible by 9 if and only if $a + b + c + d$ is divisible by 9.