MATHEMATICS COMPREHENSIVE EXAMINATION, FALL 2004

NAME: ____

Single Variable Calculus

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1. Suppose you know that $f'(t) = \sin(t^2)$ and $f(0) = 1$. Which of the following numerical methods could you use to estimate $f(2)$? Circle all that apply and briefly explain how each method that applies could be used.

   a) Euler's Method

   b) Newton's Method

   c) Simpson's Rule

2. The following table of values for the function $f(x)$ is given:

x	1	2	3	4	5
$f(x)$	3	2	0	1.5	4

   Estimate (showing all your work):

   a) $f'(2)$

   b) $\int_1^5 f(x)\,dx$ using a right-hand Reimann sum with $n = 2$ equal subdivisions

3. Suppose that f is a function which satisfies $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 0$. Circle the statement(s) below that could be true of f. Give a possible formula for $f(x)$ for each case you choose.

   a) The graph of f has an inflection point when $x = 0$.

   b) f achieves a local maximum at $x = 0$.

   c) f achieves a local minimum at $x = 0$.

   d) The graph of f is a straight line.

4. Let $f(z) = \sum_{k=0}^{\infty} z^k$ for $\frac{1}{4} \leq z \leq \frac{1}{2}$. Mark each of the following statements true or false. Show your work in each part.

   a) $\int_{1/4}^{1/2} f(z)\,dz = \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \left[\left(\frac{1}{2}\right)^{k+1} - \left(\frac{1}{4}\right)^{k+1} \right]$

   b) $\int_{1/4}^{1/2} f(z)\,dz = -\ln(2)$

5. $\int xe^x\,dx =$

6. Suppose a graphics program is written which displays a circular dot that grows and shrinks periodically. Suppose the radius, as a function of time, satisfies $r(t) = 3\cos(\pi t/4)$. Find the rate at which the area of this circular dot is changing when $t = 1$. Show your work.

7. Explain clearly why the function $f(x) = x - \frac{1}{x}$ can have at most one zero with $x < 0$ and at most one zero with $x > 0$.

8. Suppose that $\int_{-2}^{4} f(x)\,dx = 10$. Consider the following integrals. For those that can be evaluated with only the given information, give their value in the space provided. Write "NO" in the spaces for those that you can't necessarily determine the value, given only the above information.

   (a) $\int_{-2}^{4} |f(x)|\,dx =$ ____________

   (b) $\int_{4}^{-2} f(x)\,dx =$ ____________

   (c) $\int_{-2}^{4} (f(x)+2)\;dx =$ ____________

   (d) $\int_{-2}^{4} (f(x))^2\,dx =$ ____________

9. Let A be the region under the curve $y = \frac{1}{x^2}$ on the interval $[1, 4]$. Find the number a such that the line $x = a$ cuts the area of A into two pieces of equal area.

10. Suppose f and g are functions such that $\lim_{x\to 5} f(x) = 2$ and $\lim_{x\to 5} g(x) = 0$. Of the following statements, only <u>one</u> is a faulty conclusion given the information above. Circle this statement and explain why the statement is not always true.

    (a) $\lim_{x\to 5} [f(x)g(x)]$ exists.

    (b) $\lim_{x\to 5^+} f(x) = 2$

    (c) $g(5) = 0$

    (d) The graph of the function $f(x)$ has no vertical asymptote at $x = 5$.

Multivariable/Vector Calculus

In this section, the vector $\vec{v}$ with x-component a and y-component b is denoted $\vec{v} = (a, b)$.

11. Sketch the region of integration, and evaluate the integral:

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx\,dy$$

12. Find the absolute (global) maximum and minimum of $f(x, y) = 2x^2 + 2xy + y^2 + 2x$ on the region $2x^2 + 2xy + y^2 \le 2$. (This is the region inside an ellipse whose boundary includes values $-\sqrt{2} \le x \le \sqrt{2}$.) Note: In this problem, it is much easier to find a formula for f on the boundary of the region than to use Lagrange multipliers.

13. Find an equation for the plane through the point $(1, 2, 3)$ that is parallel to the vectors $\vec{u} = (1, 0, 4)$ and $\vec{v} = (-2, 1, 2)$.

14. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = (y, x+y)$, and C is the arc of the parabola $x = y^2$ from $(0,0)$ to $(4,-2)$.

15. Sketch the contour diagram for $f(x,y) = x^2 + y^2 - 1$, with level curves labeled $f = 0$, $f = 3$, and $f = 8$. Also label and mark units on your axes appropriately.

Linear Systems

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16. Fill in the blank by choosing (a), (b), or (c) and justify why your answer makes sense (do not just say something like "I remember it as a fact" or "That's the definition of blah-space").

    The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is in the ____________ of A.

    (a) row space (b) column space (c) null space

17. Let S be the subspace of $\mathbf{R}^3$ spanned by $[1,0,0]$ and $[1,0,1]$. Describe $S^\perp$ (describe it geometrically and give its dimension), and give a basis for it. Make sure you give an adequate explanation of your answer.

18. Let $\vec{v}$ and $\vec{w}$ be nonzero vectors in $\mathbf{R}^n$. Let $\vec{u} = \text{proj}_{\vec{w}}\vec{v}$, the projection of $\vec{v}$ onto $\vec{w}$. Is $\vec{u}$ necessarily a scalar multiple of only $\vec{v}$, only $\vec{w}$, both $\vec{v}$ and $\vec{w}$, or neither $\vec{v}$ nor $\vec{w}$? Explain why.

19. Find all eigenvalues and their corresponding eigenvectors for the following matrix: $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

20. Let S be the set of all vectors in $\mathbf{R}^2$ of the form $[1, y]$, $y \in \mathbf{R}$. Is S a subspace of $\mathbf{R}^2$? Verify your answer by giving the definition of "subspace" and showing whether or not S satisfies the definition.

Discrete Mathematics

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21. Find the coefficient of a^3b^5 in $(2a-b)^8$. Give your final answer as a single number.

22. a) Prove that the relation $\sim$ is an equivalence relation on $\mathbf{Z}$: $a \sim b \iff 6$ divides $(a-b)$.

    b) How many equivalence classes are there? Give a representative of each class.

23. Let A be a set of n elements. The power set, $P(A)$, of a set A is the set of all subsets of A.

    a) How many elements are in $P(A)$? Prove it.

    b) Prove or Disprove: Let A and B be two finite sets. Then $P(A \cup B) = P(A) \cup P(B)$.

24. Prove by induction for $n \geq 2$:

$$2 + 6 + 12 + \cdots + (n^2 - n) = \frac{n(n^2-1)}{3}$$

25. In this problem, p, q, and r are statements. Prove or disprove:

$$[p \to (q \to r)] \iff [(p \to q) \to (p \to r)]$$