BUCKMIRE

MATHEMATICS COMPREHENSIVE EXAMINATION, SPRING 2006

Discrete Mathematics

Please answer the following questions without working out the permutations, combinations, factorials or powers (i.e. do not simplify your answers).

21. Let $A = \{(p,q) | p, q \in \mathbf{Z},\ q \neq 0\}$. Define the relation $\sim$ on A by

$$(p,q) \sim (r,s) \iff p \cdot s = r \cdot q$$

Prove or disprove: $\sim$ is an *equivalence* relation on A.

Symmetric $a \sim b \Rightarrow b \sim a$ ✓

$(r,s) \sim (p,q) \iff r \cdot q = p \cdot s$ this is true since $ps = rq \Leftrightarrow (p,q) \sim (r,s)$

Reflexive $a \sim a$

$(p,q) \sim (p,q) \iff p \cdot q = pq$ this is true always ✓

Transitive $a \sim b$ and $b \sim c \Rightarrow a \sim c$

$(p,q) \sim (r,s)$ and $(r,s) \sim (t,u) \Rightarrow (p,q) \sim (t,u)$

$ps = rq$ and $ru = ts$ $\Rightarrow p u = t q \Leftrightarrow (p,q) \sim (t,u)$ ✓

$\frac{p}{q} = \frac{r}{s}$ and $\frac{r}{s} = \frac{t}{u}$ $\Rightarrow \frac{p}{q} = \frac{t}{u}$

So $\sim$ IS an EQUIVALENCE RELATION

22. Prove or disprove: "The following logical proposition is true no matter what the truth values of propositions p and q are: "

$$(p \to q) \to (\neg q \to \neg p).$$

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \to (\neg q \to \neg p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

✓

Tautology

23. An ice cream shop has 21 flavors of ice cream. You can order either one or two scoops on a cone.

    a. If the order in which the scoops are placed on the cone *doesn't* matter, how many different ways can you order a cone from this ice cream shop?

    b. If the order in which the scoops are placed on the cone *does* matter, how many different ways can you order a cone from this ice cream shop?

one scoop

$21 + 21^2$ = # of scoops with one or two when order matters

$21 + \binom{21}{2} + 21$ = # of scoops with one or two when order does NOT matter

choosing 2 scoops out of 21   double flavors

24. a. The following statement is false! Correct it.

    "A function $f : X \to Y$ is one-to-one if and only if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$."

    b. How many *onto* functions are there from X to Y if $|X| = 3$ and $|Y| = 2$?

a. A function $f: X \to Y$ is one-to-one if and only if $\forall x_1, x_2 \in X$ ~~such~~ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

b. Number of <u>FUNCTIONS</u> from $|X|$ to $|Y|$ is $|X|^{|Y|}$

For number of onto functions will be less.

Case 1   2 choices

x_1, x_2, x_3 (X) y_1, y_2 (Y)

All $x_i \to y_1$

OR

All $x_i \to y_2$

Range is 1 element will NOT ONTO!

CASE 2: Range is 2 elements

$x_1 \to y_1$, $x_2, x_3 \to y_2$ or $x_1, x_2 \to y_1$, $x_3 \to y_2$

3 ways   3 ways

This is ONTO case = 6 functions

25. Prove that the four-digit positive integer $n = abcd$ is divisible by 9 if and only if $a+b+c+d$ is divisible by 9.

$n = abcd$ means that $n = 1000a + 100b + 10c + d$

If abcd is divisible by 9 then

$1000a + 100b + 10c + d \equiv 0 \pmod 9$

$\equiv 999a + 99b + 9c \pmod 9$

$a+b+c+d \equiv 0 \pmod 9$

So $a+b+c+d$ is divisible by 9 also