BUCKMIRE

MATHEMATICS COMPREHENSIVE EXAMINATION, SPRING 2006

Multivariable/Vector Calculus

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11. The temperature in deg F on the surface of a 10 ft. by 8 ft. plate glass window is given by

$$T(x,y) = 72 + y + \frac{1}{80}xy, \qquad 0 \le x \le 10,\ 0 \le y \le 8.$$

   a. Find the gradient $\nabla T(x,y)$.

   b. If a fly were located at the point $(1,1)$, in which direction would it start moving if it wanted to increase its temperature the most? Your answer should be a 2-dimensional vector.

a. $\vec{\nabla} T = (T_x, T_y) = \left(\frac{1}{80}y,\ 1 + \frac{1}{80}x\right)$

b. Greatest increase is in direction of $\vec{\nabla} T$ at $(1,1)$

$\vec{\nabla} T(1,1) = \left(\frac{1}{80},\ 1 + \frac{1}{80}\right) = \begin{pmatrix} \frac{1}{80} \\ \frac{81}{80} \end{pmatrix}$

12. The function $F(x,y) = x^2 - y^3$ is differentiable at the point $(2,3)$. Find the equation for the plane tangent to the graph of F at this point.

$z = \cancel{z}\, F(a,b) + F_x(a,b)(x-a) + F_y(a,b)(y-b)$

$= F(2,3) + F_x(2,3)(x-2) + F_y(2,3)(y-3)$

$F_x = 2x \quad F_x(2,3) = 4$

$F_y = -3y^2 \quad F_y(2,3) = -3 \cdot 3^2 = -27$

$F(2,3) = 2^2 - 3^2 = \cancel{-4} = 4 - 9 = -5$

$\cancel{z = 4 +}\ z = -5 + 4(x-2) + -27(y-3)$

$z = -5 + 4x - 8 - 27y + 81$

$\boxed{-4x + 27y + z = 68}$

13. Once again, consider the temperature in deg F on the surface of a 10 ft. by 8 ft. plate glass window given by

$$T(x,y) = 72 + y + \frac{1}{80}xy, \qquad 0 \le x \le 10,\ 0 \le y \le 8.$$

Starting at the point $(10,5)$, a fly walks across the window along the curve $(x(t), y(t)) = (10-t, t+5)$ for $0 \le t \le 1$ minute.

a. At what point on this path was the temperature greatest?

b. What was the value of that greatest temperature?

$$T(t) = 72 + t + 5 + \frac{1}{80}(10-t)(t+5) \qquad 0 \le t \le 1$$

$$T'(t) = 1 + \frac{1}{80}\left((-1)(t+5) + (10-t)(1)\right)$$

$$= 1 + \frac{1}{80}(15-2t) = 0$$

$$15 - 2t = -80$$

$$-2t = -95$$

$$t = 47.5$$

NOT IN SET!

$$T(0) = 72 + 5 + \frac{1}{80}(15)$$

when $t=1$
$(x,y) = (9,6)$
Temp is greatest and equals $78 + \frac{27}{40}°F$

$$T(1) = 72 + 6 + \frac{1(9.6)}{80}$$

14. $\displaystyle\int_0^1 \int_z^3 1 + w^2 z \, dw\, dz = \int_0^1 \left[w + \frac{w^3}{3} z\right]\Big|_z^3 dz$

$$= \int_0^1 \left[3 + \frac{3^3}{3}z\right] - \left[z + \frac{z^3}{3}\cdot z\right] dz$$

$$= \int_0^1 8z + 3 - \frac{z^4}{3} dz$$

$$= 4z^2 + 3z - \frac{z^5}{15}\Big|_0^1 = 4 + 3 - \frac{1}{15} = 7 - \frac{1}{15}$$

$$= \frac{104}{105}$$

15. Consider the vector field $\vec{F}(x, y) = (x, y)$ and the curve $\vec{r}(t) = (\cos t, \sin t), 0 \leq t < 2\pi$.

a. Without calculating it, clearly explain why the line integral $\int_{t=0}^{t=2\pi} \vec{F} \cdot d\vec{r} = 0$.

b. Then confirm that $\int_{t=0}^{t=2\pi} \vec{F} \cdot d\vec{r} = 0$ by showing the details of the calculation.

a. $\vec{F} = \vec{\nabla}\phi$ where $\phi = \frac{x^2}{2} + \frac{y^2}{2}$ $\quad \vec{\nabla}\phi = (x, y) = \vec{F}$

$\oint \nabla\phi \cdot d\vec{r} = 0$ by Fundamental Theorem of line integrals

$$\int_{t=0}^{t=2\pi} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} (-\sin t \cos t + \sin t \cos t)\, dt$$

$$= \int_0^{2\pi} 0 \, dt = 0$$