Calculus 1

1. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.

$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x+1) + k, & -1 < x \leq 2 \\ 2x^3 + x + 7, & x \leq -1 \end{cases}$$

$$\boxed{\begin{matrix} K = 4 \\ m = \frac{5}{3} \end{matrix}}$$

For continuity $\lim\limits_{x \to a} f(x) = f(a)$

At $x = 2$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x^2 + 5 = 2^2 + 5 = 9$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} m(x+1) + K = m(2+1) + K = 3m + K$

$3m + K = 9$ iff f is cont at $x = 2$

At $x = -1$

$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} 2x^3 + x + 7 = 2(-1)^3 + (-1) + 7 = 4$

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} m(x+1) + K = m(-1+1) + K = K$

$K = 4$ iff f is cont at $x = -1$

2. Use the limit laws, and if necessary, L'Hôpital's Rule to find the following limit

$$\lim_{x \to 0^+} (1 + 2x)^{-3/x}$$

Let $p = (1 + 2x)^{-3/x}$

$\ln p = \ln\left[(1+2x)^{-3/x}\right] = -\frac{3}{x} \ln(1 + 2x)$

$\lim\limits_{x \to 0^+} \ln p = \lim\limits_{x \to 0^+} \frac{-3\ln(1+2x)}{x} \overset{L'H}{=} \lim\limits_{x \to 0^+} \frac{-\frac{3}{1+2x} \cdot 2}{1} = -6$

$\lim\limits_{x \to 0^+} e^{\ln p} = \lim\limits_{x \to 0^+} p = e^{\lim_{x \to 0^+} \ln p} = e^{-6}$

[illegible]

3. Suppose that the number of bacteria in a culture at time t is given by

$$N = 5000(25 + te^{-t/20}).$$

(a) Find the largest and smallest number of bacteria in the culture during the time interval $0 \le t \le 100$.

$$\frac{dN}{dt} = 5000 \cdot e^{-t/20} + 5000t \cdot -\frac{1}{20} e^{-t/20}$$

$$N' = 5000e^{-t/20}\left[1 - \frac{t}{20}\right]$$

When $t = 20$, $N' = 0$

$N(0) = 5000 \cdot 25 = 125000$ $\quad N(20) = 5000(25 + 20e^{-1})$

$N(100) = 5000 \cdot (25 + 100e^{-5})$

(b) At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?

$N(100) = 125000 + 500000e^{-5}$ ← global max

$N(20) = 125000 + 100000e^{-1}$

$N(0) = 125000$ ← global min

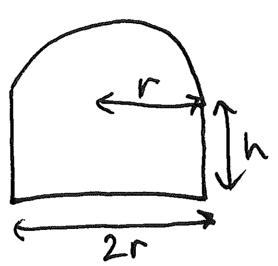
(b) N' is most negative at largest + value of N''

i.e. at $t = 40$

$$N'' = 5000\left\{-\frac{1}{20}e^{-t/20} \cdot \left(1 - \frac{t}{20}\right) + e^{-t/20} \cdot -\frac{t}{20}\right\} = \frac{5000e^{-\frac{t}{20}}}{-20}\left[2 - \frac{t}{20}\right]$$

$N'' = 0 \Leftrightarrow t = 40$

4. A church window consisting of a rectangle topped by a semicircle is to have a perimeter p. Find the radius of the semicircle if the area of the window is to be maximum.



$$p = 2r + 2h + \pi r \Rightarrow h = \frac{p - 2r - \pi r}{2}$$

$$A = 2rh + \frac{\pi r^2}{2}$$

$$A = 2r\left(\frac{p - 2r - \pi r}{2}\right) + \frac{\pi r^2}{2}$$

$$= r(p - 2r - \pi r) + \frac{\pi r^2}{2}$$

$$= pr - 2r^2 - \pi r^2 + \frac{\pi r^2}{2}$$

$$A = pr - 2r^2 - \frac{\pi r^2}{2}, \quad 0 \le r \le p$$

$$\frac{dA}{dr} = p - 4r - \pi r = 0$$

$$\Rightarrow p = (4 + \pi)r$$

$$\Rightarrow r = p/(4 + \pi)$$

$$\frac{d^2A}{dr^2} = -4 - \pi < 0$$

5. Suppose that you have money in an account that is earning interest at an APR of 4% compounded continuously and that you add a total of \$1000 to the account every year applied at a constant rate so that the rate of change of money M in the account is given by

$$\frac{dM}{dt} = (.04)M + 1000,$$

where time t is measured in years and money M is measured in dollars. Also suppose that you have \$8000 in the account at the start of year three, i.e. $M(3) = 8000$.

(a) Use a local linear approximation to estimate how much money will be in the account at the end of January of the third year, i.e. use a local linear approximation to estimate $M(3 + \frac{1}{12})$.

(b) Use the second derivative to determine if your approximation is an overestimate or an underestimate. Explain your answer.

$M(3) = 8000$

$M'(3) = .04 \cdot 8000 + 1000$
$= 320 + 1000$
$= 1320$

$$M(3+\tfrac{1}{12}) \approx M(3) + M'(3) \cdot \tfrac{1}{12}$$
$$\approx 8000 + 1320 \cdot \tfrac{1}{12}$$
$$= 8000 + 110$$
$$\approx 8110$$

$$M'' = 0.04M' = 0.04^2 M + 40$$
$$M''(3) = 0.04M'(3) = (0.04)\cdot(1320) = \frac{5280}{100} = 52.8 > 0$$

since $M'' > 0$ at 3, $M(3+\frac{1}{12})$ will be an under-estimate