PRACTICE Comprehensive Exam **Student No.** (AXXXXXXXX): ________

Department of Mathematics Tuesday, March 15, 2016

Closed book. Closed notes. NO CALCULATORS. Time allowed: 3 hours for 5 sections (proportionally less if taking fewer than 5 sections). In other words, 36 minutes for each section taken. Please write very legibly and cross out all scratch work.

Calculus 1

1. ________ 2. ________ 3. ________ 4. ________ 5. ________ Total: ________

Calculus 2

6. ________ 7. ________ 8. ________ 9. ________ 10. ________ Total: ________

Multivariable Calculus

11. ________ 12. ________ 13. ________ 14. ________ 15. ________ Total: ________

Linear Algebra

16. ________ 17. ________ 18. ________ 19. ________ 20. ________ Total: ________

Discrete Mathematics

21. ________ 22. ________ 23. ________ 24. ________ 25. ________ Total: ________

Discrete Mathematics

21. Let R be a relation on the set $\mathbb{R}^2$ defined by

$$(x_1, y_1)R(x_2, y_2) \text{ if and only if } x_1 - y_1 = x_2 - y_2.$$

(a) (2 points) Prove that R is an equivalence relation on $\mathbb{R}^2$.

(b) (1 point) Describe the equivalence class of the element $(3, 2)$ both in set-builder notation and geometrically.

(c) (1 point) Describe geometrically (provide a geometric interpretation of) how the equivalence classes of R partition the plane $\mathbb{R}^2$.

22. Prove the following is a tautology for statements p, q and r:

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

(HINT: Use a truth table!)

23. Consider the letters (or characters) from the standard 26-member English alphabet.

(a) What is the number of character strings containing six letters?

(b) What is the number of character strings containing six letters if no letter gets repeated in each string?

(c) What is the number of six-letter strings with exactly two vowels (where the set of vowels in English is defined to be the 7 letters $\{a, e, i, o, u, w, y\}$)?

(d) What is the number of six-letter strings with the letter a?

(HINT: You do not need to numerically simplify your calculations.)

24. Let A be a set of n elements. The power set of A, $P(A)$ is the set of all subsets of A.

   (a) How many elements are there in $P(A)$? Prove it.

   (b) Prove or disprove: $P(A \cup B) = P(A) \cup P(B)$.

25. Prove by induction for all integers $n \geq 2$:

$$2 + 6 + 12 + \cdots + (n^2 - n) = \frac{n(n^2-1)}{3}.$$

More Discrete Mathematics Practice Questions

1. Prove that these four statements about the integers n are equivalent:
   (a) n^2 is odd;
   (b) $1 - n$ is even;
   (c) n^3 is odd;
   (d) $n^2 + 1$ is even.

2. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

3. Define the term **bijection** and then provide a graphical representation of a bijection from a finite set A to B.

4. Let R be a relation on the set $\mathbb{R}^2$ defined by

$$(x_1, y_1)R(x_2, y_2) \text{ if and only if } x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

   (a) Prove that R is an equivalence relation on $\mathbb{R}^2$.
   (b) Describe the equivalence class of the element $(3, 2)$ both in set-builder notation and geometrically.
   (c) Describe geometrically how the equivalence classes of R partition the plane $\mathbb{R}^2$.

5. Let

$$\begin{aligned} A &= \{x \in \mathbb{Z} : \ 4x \equiv 19 \pmod{21}\} \\ B &= \{x \in \mathbb{Z} : \ x \equiv 10 \pmod{21}\} \\ C &= \{x \in \mathbb{Z} : \ 3x \equiv 2 \pmod{7}\}. \end{aligned}$$

   (a) Prove that $A = B$.
   (b) Prove that $A \subseteq C$.

6. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property that $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

7. (a) Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is one-to-one but not onto. Very briefly explain why your example is not onto (but no need to prove it is one-to-one).
   (b) Prove or give a counterexample: Let f be a function from $\mathbb{N}$ to $\mathbb{N}$; if f is onto, then it is a bijection.

8. Show that these statements about the real number x are equivalent:
   (i) x is rational,
   (ii) $x/2$ is rational,
   (iii) $3x - 1$ is rational.