PRACTICE Comprehensive Exam **Student Number (AXXXXXXXX)**: ________
Department of Mathematics Tuesday, March 15, 2016

Closed book. Closed notes. NO CALCULATORS. Time allowed: 3 hours for 5 sections (proportionally less if taking fewer than 5 sections). In other words, 36 minutes for each section taken. Please write very legibly and cross out all scratch work.

Calculus 1

1. ________ 2. ________ 3. ________ 4. ________ 5. ________ Total: ________

Calculus 2

6. ________ 7. ________ 8. ________ 9. ________ 10. ________ Total: ________

Multivariable Calculus

11. ________ 12. ________ 13. ________ 14. ________ 15. ________ Total: ________

Linear Algebra

16. ________ 17. ________ 18. ________ 19. ________ 20. ________ Total: ________

Discrete Mathematics

21. ________ 22. ________ 23. ________ 24. ________ 25. ________ Total: ________

Linear Algebra

16. Decide if each of the statement is TRUE or FALSE. If FALSE, explain why its false or give a counter example that demonstrates why the statement is false. If TRUE, provide a short proof demonstrating why the statement is true. In all cases A is an $n \times n$ matrix of real numbers.

   (a) $\det(-A) = -\det(A)$.

   (b) For $A\vec{x} = \vec{b}$ to have a solution, $\vec{b}$ must be in the rowspace of A.

   (c) A is not invertible if and only if 0 is an eigenvalue of A.

   (d) $(A^T)^T = A$.

17. Consider the following vector $\vec{v}$ and subspace $\mathcal{W}$ in $\mathbb{R}^4$ given below:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathcal{W} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

(HINT: Recall that the projection matrix P onto a subspace spanned by the columns of a matrix A is given by $P = A(A^T A)^{-1} A^T$.)

(a) Find the orthogonal decomposition of $\mathbf{v}$ with respect to W, i.e. find vectors $\mathbf{v_1} \in \mathcal{W}$ and $\mathbf{v_2} \in \mathcal{W}^{\perp}$ such that $\mathbf{v} = \mathbf{v_1} + \mathbf{v_2}$.

(b) Confirm that your choice of $\mathbf{v_1}$ and $\mathbf{v_2}$ are indeed orthogonal and add up to $\mathbf{v}$.

18. (a) Write down the definition of the term "linearly independent."

    (b) If vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly independent, will $\vec{u} + \vec{v}, \vec{v} + \vec{w}$, and $\vec{u} + \vec{w}$ also be linearly independent? Justify your answer.

19. Consider

$$A = \begin{bmatrix} 6 & 4 \\ -6 & -4 \end{bmatrix}$$

(a) Confirm that $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors for A.

(b) By diagonalizing A, show that an expression for $A^n = \begin{bmatrix} 3(2^n) & 2^{n+1} \\ -3(2^n) & -2^{n+1} \end{bmatrix}$.

20. Given

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

(a) Find the rank of A.

(b) Find bases for $\text{col}(A)$, $\text{row}(A)$ and $\text{null}(A)$.

More Linear Algebra Practice Questions

1. Consider a vector $\mathbf{v}$ and subspace $\mathcal{W}$ given below:

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \quad \mathcal{W} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

   Find the orthogonal decomposition of $\mathbf{v}$ with respect to W.

2. Consider the planes $x + y + 4z = 5$ and $x + 2y + 3z = 4$ in $\mathbb{R}^3$.

   (a) Find the equation of the line corresponding to their intersection and write it in parametric and vector form.

   (b) Is the point $(1, 0, 1)$ on the line of intersection of the two planes? How about the origin? EXPLAIN YOUR ANSWERS.

3. Find the eigenvalues and eigenspaces of $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

4. For what values of k does the following linear system have no solution?

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

5. Find the minimum distance between the following two parallel planes (Do not use a formula, show all work.)

$$\begin{aligned} 2x - y + z &= 1 \\ -4x + 2y - 2z &= 1 \end{aligned}$$

6. The system of equations $A\vec{x} = \vec{b}$ must have ____________, ____________ or ____________ number of solutions. Provide an example of a linear system which demonstrates each case.

7. **Prove or provide a counter-example.** Let A be a square invertible matrix.

$$(A^{-1})^T = (A^T)^{-1}$$

8. **Prove or provide a counter-example.** Let A be a $m \times n$ matrix. The system of equations $A\vec{x} = \vec{b}$ is consistent (i.e. has at least 1 solution) if and only if $\vec{b}$ is in the column space of A.

9. Consider the planes $x + y + 4z = 5$ and $x + 2y + 3z = 4$ in $\mathbb{R}^3$.

   (a) Find the equation of the line corresponding to their intersection and write it in parametric and vector form.

   (b) Is the point $(1, 0, 1)$ on the line of intersection of the two planes? How about the origin? EXPLAIN YOUR ANSWERS.

10. TRUE or FALSE. Give a reason for your answer.

   (a) $\det(A)$ is given by the sum of its eigenvalues.

   (b) The nullspace of an $m \times n$ matrix is always a subspace of $\mathbb{R}^m$

   (c) The rowspace of an $m \times n$ matrix is always a subspace of $\mathbb{R}^n$

   (d) The eigenvalues of A and A^T are always identical.

11. (Problem 48 from Section 2.3 of Poole **2nd Edition**)
Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$, and let $\vec{v}$ be a vector in $\mathbb{R}^n$. Suppose that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ with $c_1 \neq 0$. Prove that $\{\vec{v}, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

12. (Problem 42 from Section 3.3 of Poole **2nd Edition**) Let $\mathcal{O}$ by the zero matrix (all entries are zero).

   (a) Prove that if A is invertible and $AB = \mathcal{O}$, then $B = \mathcal{O}$.

   (b) Give a counterexample to show that the result in part (a) may fail if A is not invertible.