6 Computing Derivatives the Quick and Easy Way

In the previous section we talked about what the derivative is, and we computed several examples, and then we got quite tired of computing those examples over and over. In this section we'll come up with some techniques to make computation of derivatives easier. It corresponds roughly to sections 2.3 and 2.4 in your textbook.

6.1 Basics: The simple rules

1. If c is a constant and $f(x) = c$ then $f'(x) = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

   Conceptually, a constant function never changes, so the rate of change is 0.

   Geometrically, a constant function is a horizontal line; thus we think of the slope everywhere as being 0.

   Example 6.1. $(3^{3^{3^3}})' = 0$.

2. If $f(x) = x$, then $f'(x) = 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} 1 = 1.$$

   Conceptually, if we have the "identity" function, then whenever we change the input then the output should change by exactly the same amount. Thus the rate of change is 1.

   Geometrically, this is a line with slope 1.

3. If c is a constant and g is a function and $f(x) = c \cdot g(x)$, then $f'(x) = c(g'(x))$.

$$f'(x) = \lim_{h\to 0} \frac{cg(x+h) - cg(x)}{h} = c \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = c \cdot g'(x).$$

   Conceptually, if changing x by a bit changes $g(x)$ by a certain amount, then it will change $2g(x)$ by twice that amount–multiplying by a scalar should just change the rate of change by the same amount everywhere.

   Geometrically, multiplying by a constant is just stretching vertically–and all the slopes will be stretched by that same amount.

Example 6.2. If $f(x) = 5x$ then $f'(x) = (5 \cdot x)' = 5 \cdot x' = 5$.

4. If f and g are functions then $(f+g)'(x) = f'(x) + g'(x)$.

$$\begin{aligned}(f+g)'(x) &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).\end{aligned}$$

Conceptually, if changing the input by a bit changes f by a certain amount and g by a different amount, then it changes $f+g$ by the sum of those two amounts–figure out how much it changes each part and then add them together to find out how much it changes the whole.

Geometrically, if we add two functions together it's just like stacking them on top of one another, so the slope at any point will be the sum of the slopes.

Example 6.3. Let $f(x) = 3x - 7$. Then $f'(x) = (3x)' - 7' = 3(x') - 0 = 3$.

This rule is really important but so far we can't do much with it–we don't have quite enough rules yet.

5. (Power Rule) If $f(x) = x^n$ where n is a positive integer, then $f'(x) = nx^{n-1}$. In fact, if $g(x) = x^r$ and r is any real number, then $g'(x) = rx^{r-1}$. We'll only prove this for integers, using the difference-of-nth-powers rule.

$$\begin{aligned}f'(x) &= \lim_{z \to x} \frac{z^n - x^n}{z - x} = \lim_{z \to x} \frac{(z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z - x} \\ &= \lim_{z \to x} z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1} = x^{n-1} + \cdots + x^{n-1} = nx^{n-1}.\end{aligned}$$

Now that we have this, we can compute all sorts of derivatives.

Example 6.4.
- $(x^2+1)' = 2x + 0 = 2x$.
- $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$.
- $(\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$.
- $(3\sqrt{x} + x^5 - 7)' = \frac{3}{2\sqrt{x}} + 5x^4 + 0$.

6. (Product Rule) If f and g are functions then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

   Conceptually, we sort of know this already; if we add a bit on to f and a bit on to g, then we get $(f + f_h)(g + g_h) = fg + fg_h + gf_h + g_h f_h$, and in the limit we can treat $g_h f_h$ as being zero. So this is the same as multiplying the bit we add to g with f, and multiplying the bit we add to f with g, and then adding the two.

   Example 6.5. $((3x - 2)(x - 1))' = (3x^2 - 5x + 2)' = 6x - 5$.

   Alternatively, $((3x-2)(x-1)' = (3x-2)'(x-1)+(3x-2)(x-1)' = 3\cdot(x-1)+1\cdot(3x-2) = 6x - 5$.

   This rule isn't terribly important as long as we're only working with rational functions. Once we include anything else, like trig functions, it is critical.

   Remark 6.6. We can get the power rule from the product rule instead of trying to get it directly.

7. (Quotient Rule): If f and g are functions then

$$(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

$$\begin{aligned}
(f/g)'(x) &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left(\lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} + \lim_{h\to 0} \frac{f(x)g(x) - f(x)g(x+h)}{h} \right) \\
&= \frac{1}{g(x)^2} \left(g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

Example 6.7.
- $\left(\frac{x-1}{x^3}\right)' = (x^{-2} - x^{-3})' = -2x^{-3} + 3x^{-4}$.

  Alternatively,

$$\left(\frac{x-1}{x^3}\right)' = \frac{(x-1)'x^3 - (x-1)3x^2}{x^6} = \frac{x^3 - 3x^3 + 3x^2}{x^6} = -2x^{-3} + 3x^{-4}.$$

- $\left(\frac{2+3x}{3-5x}\right)' = \frac{(2+3x)'(3-5x) - (2+3x)(3-5x)'}{(3-5x)^2} = \frac{9 - 15x + 10 + 15x}{(3-5x)^2} = \frac{19}{(3-5x)^2}$

6.2 Trigonometric derivatives

We cannot neglect the trigonometric functions—no matter how much we might wish to on occasion. All of the rules for trigonometric derivatives rely on what are known as the *angle addition formulas*:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Note: you probably won't ever need to know these formulas again in this class. But I will need them for another page or so of these notes.

Using this we can compute

1.
$$\begin{aligned}
(\sin(x))' &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \left(\lim_{h\to 0} \frac{\sin(h)\cos(x)}{h}\right) + \left(\lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h}\right) \\
&= \cos(x) \lim_{h\to 0} \frac{\sin h}{h} + \sin(x) \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= \cos(x) + \sin(x) \lim_{h\to 0} \frac{\cos^2(h) - 1}{h(\cos(h)+1)} \\
&= \cos(x) - \sin(x) \lim_{h\to 0} \frac{\sin^2(h)}{h(\cos(h)+1)} \\
&= \cos(x) - \sin(x) \left(\lim_{h\to 0} \frac{\sin(h)}{\cos(h)+1}\right)\left(\lim_{h\to 0} \frac{\sin h}{h}\right) \\
&= \cos(x) - \sin(x) \cdot 0 \cdot 1 = \cos(x).
\end{aligned}$$

2. A similar argument shows that $(\cos(x))' = -\sin(x)$.

Further using the product and quotient rules, we observe that

- $$(\tan(x))' = \left(\frac{\sin x}{\cos x}\right)' \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

- $$(\cot(x))' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

- $$(\sec(x))' = \left(\frac{1}{\cos x}\right)' = \frac{0 + \sin x}{\cos^2(x)} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \sec(x)\tan(x)$$

- $$(\csc(x))' = \left(\frac{1}{\sin x}\right)' = \frac{0 - \cos(x)}{\sin^2(x)} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\csc(x)\cot(x).$$

Remember that as long as you know the derivatives of sin and cos you can always compute these four derivatives whenever you need them.

Example 6.8.

1. If $f(t) = 3\sin t + \cos t$, then $f'(t) = 3\cos t - \sin t$.
2. Find the tangent line to $y = 6\cos x$ at $(\pi/3, 3)$.

   We see that $y' = -6\sin x$, and thus when $x = \pi/3$ we have $y' = -3\sqrt{3}$. Recalling that the equation of our line is $y = m(x - x_0) + f(x_0)$, we have the equation $y = -3\sqrt{3}(x - \pi/3) + 3$.
3. IF $g(\theta) = \theta \sin\theta + \frac{\cos\theta}{\theta}$, then
   $$g'(\theta) = (\sin\theta + \theta\cos\theta) + \frac{-\theta\sin\theta - \cos\theta}{\theta^2}.$$
4. If $h(x) = \frac{x}{2 - \tan x}$, then
   $$h'(x) = \frac{(2 - \tan x) + x\sec^2 x}{(2 - \tan x)^2}.$$
5. We can also compute second derivatives. $\sin'' x = -\sin x$. $\cos'' x = -\cos x$.
   $$\tan'' x = (\sec x \sec x)' = \sec x \tan x \sec x + \sec x \tan x \sec x = 2\sec^2 x \tan x.$$

6.3 The Chain Rule

To start with an example, suppose $g(x) = (\sin x)^2$. Then

$$g'(x) = ((\sin x)(\sin x))' = \cos x \sin x + \cos x \sin x = 2 \sin x \cos x.$$

Remembering that $(x^2)' = 2x$, we notice that this looks suggestive. It also leads us to ask what happens when we build up functions by composition, that is, plugging one function into another, as we have here.

If we want to freely build complex functions from simple ones, we need to be able to combine them in chains. Remember that we define the function $f \circ g$ by $(f \circ g)(x) = f(g(x))$; we take our input x, plug it into g, and then take the output $g(x)$ and plug it into f.

We can see how this is useful in two different ways. First, as we saw earlier, it lets us build up functions.

1. $(x+1)^2 = (f \circ g)(x)$ where $g(x) = x+1$ and $f(x) = x^2$.

2. $(x^2+1)^2 = (f \circ g)(x)$ where $g(x) = x^2+1$ and $f(x) = x^2$.

3. $\sin^2(x) = (f \circ g)(x)$ where $g(x) = \sin x$ and $f(x) = x^2$.

Second, sometimes composition of functions really is the best way to describe what's going on, especially when you have a "causal chain" where one process causes a second which causes a third. For instance, suppose you're driving up a mountain at 2 km/hr, and the temperature drops 6.5° C per kilometer of altitude. You can think about your temperature as a function of your height, which is itself a function of the time; then the numbers I gave you are the rates of change, or derivatives, of each function.

It's not that hard to convince yourself that you'll get colder by about 13° C per hour. Does this work in general?

Proposition 6.9 (Chain Rule)**.** *Suppose f and g are functions, such that g is differentiable at a and f is differentiable at $g(a)$. Then $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.*

Proof.

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{h \to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h} \\
&= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h} \\
&= \left(\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \right) \left(\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \right) \\
&= f'(g(a)) \cdot g'(a).
\end{aligned}$$

□

Remark 6.10. 1. When we write $f'(g(x))$, we mean the function f' evaluated at the point $g(x)$, or in other words, the derivative of f at the point $g(x)$.

2. It can be helpful as a way of remembering the chain rule that

$$\frac{d(f \circ g)}{dx} = \frac{d(f \circ g)}{dg} \cdot \frac{dg}{x}.$$

Don't take this too seriously as actively meaning anything, since it only sort of does, but it's quite helpful for the memory.

Example 6.11. 1. $(x+1)^2 = (f \circ g)(x)$ where $g(x) = x+1$ and $f(x) = x^2$. Then $f'(x) = 2x$ and $g'(x) = 1$, so

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(g(x)) \cdot 1 = 2(x+1) \cdot 1 = 2x + 2.$$

Sanity check:

$$(f \circ g)'(x) = (x^2 + 2x + 1)' = 2x + 2.$$

2. $(x^2+1)^2 = (f \circ g)(x)$ where $g(x) = x^2 + 1$ and $f(x) = x^2$. Then $f' = 2x$, $g' = 2x$, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(g(x)) \cdot 2x = 2(x^2+1) \cdot 2x = 4x^3 + 4x.$$

Sanity check:

$$(f \circ g)'(x) = (x^4 + 2x^2 + 1)' = 4x^3 + 4x.$$

3. $\sin^2(x) = (f \circ g)(x)$ where $g(x) = \sin x$ and $f(x) = x^2$. Then $f'(x) = 2x$, $g'(x) = \cos x$, and we have

$$(f \circ g)'(x) = 2(g(x)) \cdot \cos x = 2(\sin x)\cos x.$$

4. $\cos(3x) = (f \circ g)(x)$ where $f(x) = \cos(x)$ and $g(x) = 3x$. Then $f'(x) = -\sin(x)$ and $g'(x) = 3$ and

$$(f \circ g)'(x) = -\sin(3x) \cdot 3.$$

5. $\sin(x^2) = (f \circ g)(x)$ where $f(x) = \sin(x)$ and $g(x) = x^2$. Then $f'(x) = \cos x, g'(x) = 2x$, and

$$(f \circ g)'(x) = \cos(g(x)) \cdot 2x = 2x\cos(x^2).$$

6. If $f(x)$ is any function, then we can write $(f(x))^r$ as $(g \circ f)(x)$ where $g(x) = x^r$. Then

$$\frac{d}{dx}(f(x)^r = (g \circ f)'(x) = r(f(x))^{r-1} \cdot f'(x).$$

7. The derivative of $\sec(5x)$ is $\sec(5x)\tan(5x)5$.

8. What is the derivative of $\frac{1}{\sqrt[3]{x^4-12x+1}}$? We can view this as $(x^4 - 12x + 1)^{-1/3}$, and using the chain rule, we have

$$\frac{d}{dx}\frac{1}{\sqrt[3]{x^4 - 12x + 1}} = \frac{-1}{3}(x^4 - 12x + 1)^{-4/3} \cdot (4x^3 - 12).$$

9. What is the derivative of $\sec^2(x)$? By the chain rule this is $2 \cdot \sec(x) \cdot \sec'(x) = 2\sec(x) \cdot \sec(x)\tan(x) = 2\sec^2(x)\tan(x)$.

10. What is the derivative of $\tan^4(x)$? We get $4\tan^3(x)\sec'(x) = 4\tan^3(x)\sec(x)\tan(x) = 4\tan^4(x)\sec(x)$.

11. Sometimes we have to nest the chain rule. What is the derivative of $\sqrt{x^3+\sqrt{x^2+1}}$? We can pull this apart slowly.

$$\begin{aligned}\frac{d}{dx}\sqrt{x^3+\sqrt{x^2+1}} &= \frac{1}{2}(x^3+\sqrt{x^2+1})^{-1/2}\cdot\left(\frac{d}{dx}\left(x^3+\sqrt{x^2+1}\right)\right)\\ &= \frac{1}{2\sqrt{x^3+\sqrt{x^2+1}}}\left(3x^2+\frac{1}{2}(x^2+1)^{-1/2}\cdot\left(\frac{d}{dx}x^2+1\right)\right)\\ &= \frac{3x^2+\frac{2x}{2\sqrt{x^2+1}}}{2\sqrt{x^3+\sqrt{x^2+1}}}\end{aligned}$$

6.4 Barnburners

As we saw, the chain rule can stack, or chain together. As functions get more complicated we will have to use multiple applications of the product rule, quotient rule, and chain rule to pull our derivative apart.

Poll Question 6.4.1. Find

$$\frac{d}{dx}\sec(x^2+\sqrt{x^3+1}).$$

$$\begin{aligned}\frac{d}{dx}\sec(x^2+\sqrt{x^3+1}) &= \sec(x^2+\sqrt{x^3+1})\cdot\tan(x^2+\sqrt{x^3+1})\cdot(x^2+\sqrt{x^3+1})'\\ &= \sec(x^2+\sqrt{x^3+1})\cdot\tan(x^2+\sqrt{x^3+1})\cdot(2x+\frac{1}{2}(x^3+1)^{-1/2}\cdot 3x^2)\end{aligned}$$

Poll Question 6.4.2. Find

$$\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1}$$

$$\begin{aligned}\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1} &= \frac{(\sin(x^2)+\sin^2(x))'(x^2+1)-2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2}\\ &= \frac{(\cos(x^2)\cdot 2x+2\sin(x)\cos(x))(x^2+1)-2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2}.\end{aligned}$$

We can keep going with increasingly complicated problems, basically until we get bored. These are really good practice for making sure you understand how the rules fit together.

Example 6.12. Find

$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}}$$

$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}} = \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2} \cdot \left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)'$$
$$= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2} \cdot \frac{\frac{1}{2}x^{-1/2}(\cos x+1)^2 - 2(\cos x+1)(-\sin x)(\sqrt{x}+1)}{(\cos x+1)^4}$$

Example 6.13. Calculate

$$\frac{d}{dx}\left(\frac{\sin^2\left(\frac{x^2+1}{\sqrt{x-1}}\right)+\sqrt{x^3-2}}{\cos(\sqrt{x^2+1}+1)-\tan(x^4+3)}\right)^{5/3}$$