8 Inverse Functions and Exponentials

8.1 Inverse Functions

Remember we started out by saying that a function is a process: it takes an input and returns an output. Sometimes we want to undo this process. This is in fact a natural question; "What do I have to do if I want to get X" is a pretty common thought process. So our goal is: given a function f, given $f(x)$, can we find x?

Definition 8.1. If f is a function and $(g \circ f)(x) = x$ for every x in the domain of f, then we say g is an *inverse* of f.

Example 8.2.

- If $f(x) = x$ then $g(y) = y$ is an inverse to f.
- If $f(x) = 5x + 3$ then $g(y) = (y - 3)/5$ is an inverse to f.
- If $f(x) = x^3$ then $g(y) = \sqrt[3]{y}$ is an inverse to f.

Remark 8.3. A given function f has at most one inverse—if f has an inverse at all, then that means "for any y, find the x where $f(x) = y$" is a well-defined rule.

If g is an inverse to f, then the domain of g is the image of f and the domain of f is the image of g.

Unfortunately, we can't always find these inverses. For instance, if you know that $x^2 = 9$, you don't know for sure what x is: it could be 3 or -3. Similarly, if you know $\sin(x) = 0$, then x could be $n\pi$ for any integer n. The fundamental problem here is that there are some outputs that are generated by more than one input.

Definition 8.4. A function f is *1-1* or *one-to-one* (or *injective*) if, whenever $f(a) = f(b)$, we know that $a = b$.

Example 8.5. Functions which are 1-1:

- $f(x) = x$. If $f(a) = f(b)$ then $a = b$ by definition.
- $f(x) = x^3$. If $f(a) = f(b)$ then $a^3 = b^3$, and then $(a/b)^3 = 1$ so $a/b = 1$ and $a = b$.
- $f(x) = \sqrt{x}$. If $f(a) = f(b)$ then $\sqrt{a} = \sqrt{b}$ so $|a| = |b|$. But $a, b \geq 0$ since they're in the domain of f, and thus $a = b$.

Functions which are not 1-1:

- $f(x) = x^2$, since $f(-1) = f(1)$.
- $f(x) = |x|$, since $f(-2) = f(2)$.
- $\sin(x)$, since $\sin(0) = \sin(\pi)$.
- $f(x) = 3$, since $f(a) = f(b) = 3$ for any real numbers a and b.

However, we can often force a function to be one-to-one by restricting its domain.

Example 8.6.
- The function $f(x) = x^2$ on the domain $[0, +\infty)$ is 1-1. If $f(a) = f(b)$ then $a^2 = b^2$ so $a = \pm b$. But both $a, b \geq 0$ so $a = b$.
- The function $\sin(x)$ is 1-1 on the domain $[-\pi/2, \pi/2]$. If we look at the unit circle, we see that as x varies from $-\pi/2$ to $\pi/2$, the y coordinate on the unit circle is always increasing, and so never repeats itself.

This might lead us to think graphically about what the idea of 1-1-ness means:

Proposition 8.7 (Horizontal Line Test)**.** *A function f is 1-1 if and only if any horizontal line will intersect its graph in at most one point.*

It's reasonably clear that every function with an inverse must be one-to-one, since otherwise there's not a unique answer to the inverse question. Less obvious is that being 1-1 is enough to be invertible.

Proposition 8.8. *If f is a 1-1 function with domain A and image B, then there is a function f^{-1} with domain B and image A which is an inverse to f.*

We can find this inverse by writing the equation $y = f(x)$ and solving for x as a function of y. Finding an inverse for f is also a good way to prove that f is one-to-one.

Example 8.9. Let $f(x) = x^4$ with domain $(-\infty, 0]$. Then we have $y = x^4 \Rightarrow x = \pm\sqrt[4]{y}$. But we know that $x < 0$ so $x = -\sqrt[4]{y}$, and thus $g(y) = -\sqrt[y]{y}$ is an inverse for f.

Graphically, the graph of f^{-1} looks like the graph of f flipped acroos the line $y = x$, which makes sense, since a point (x, y) on the graph of f should correspond to a point (y, x) on the graph of f^{-1}. In fact, the Horizontal Line Test mentioned earlier is basically the Vertical Line Test applied to the inverse function.

Example 8.10. Take $f(x) = x^3 - x$. This function is clearly not one-to-one, since $f(1) = f(0) = f(-1) = 0$. But we can split it up into intervals where it is one-to-one. Looking at the graph, it seems natural to split it up at the critical points. And this suggests we should use calculus to study our inverse function problem.

8.1.1 Calculus of inverse functions

Now that we understand inverse functions as functions, we'd like to see what calculus can tell us about them.

Proposition 8.11. *If f is one-to-one and continuous at a, then f^{-1} is continuous at $f(a)$.*

If f is one-to-one and continuous, then f^{-1} is continuous.

We'd really like to know about the derivatives of inverse functions. We can work out what they are with some quick sketched arguments, and then can prove the answer rigorously once we know what we're looking for.

First, the argument by "it looks nice in the notation": we can rephrase this theorem as saying that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Second, if we already know that both functions are differentiable, we can use implicit differentiation:

$$\begin{aligned} f^{-1}(f(x)) &= x \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'(f(x)) &= \frac{1}{f'(x)}. \end{aligned}$$

Writing $x = f^{-1}(a)$, or equivalently $a = f(x)$, gives our statement.

Theorem 8.12 (Inverse Function Theorem)**.** *If f is a one-to-one differentiable function, and $f'(f^{-1}(a)) \neq 0$, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.*

Proof. Set $y = f^{-1}(x)$ and $b = f^{-1}(a)$. Then

$$\begin{aligned} (f^{-1})'(a) &= \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} \\ &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f(a))}. \end{aligned}$$

□

Example 8.13. Let $f(x) = x^n$ on $[0, +\infty)$; then $f^{-1}(x) = \sqrt[n]{x}$. Our formula gives

$$\begin{aligned}(f^{-1})'(a) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt[n]{(a)})} \\ &= \frac{1}{n(\sqrt[n]{(a)})^{n-1}} = \frac{1}{na^{(n-1)/n}} = \frac{1}{n}a^{(1-n)/n} = \frac{1}{n}a^{\frac{1}{n}-1}.\end{aligned}$$

Though at first this didn't look like our original answer, it is the same as the formula we had before.

Example 8.14. Let $f(x) = \sqrt[3]{5x^2+7}$. What is $(f^{-1})'(3)$?

Well, we have $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. We know that $f'(x) = \frac{1}{3}(5x^2+7)^{-2/3} \cdot 10x$, and we can work out that $f(2) = \sqrt[3]{20+7} = 3$ (by plugging in small integers until one works). Thus $f^{-1}(3) = 2$, and so we have

$$(f^{-1})'(3) = \frac{1}{\frac{1}{3}(27)^{-2/3} \cdot 20} = \frac{3 \cdot 9}{20} = \frac{27}{20}.$$

8.2 The exponential and the logarithm

Back in the first weeks of the course, we discussed the exponential functions. It's simple to define x^n when n is a positive integer, as $x \cdot x \cdot \cdots \cdot x$. It's now clear that we defined $x^{1/n}$ as the inverse function to x^n, with domain restricted to positive numbers in the case n is even and thus x^n is not one-to-one. But can we make sense of x^r where r is any real number? What would it mean to write $2^{\sqrt{2}}$?

The answer would presumably be between 2 and 4. And also between $2^{1.4}$ and $2^{1.5}$. And between $2^{1.41}$ and $2^{1.42}$. In fact, this is how we will define $2^{\sqrt{2}}$. It turns out that there will be exactly one number greater than $2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, \ldots$ and less than $2^2, 2^{1.5}, 2^{1.42}, 2^{1.415}, 2^{1.4143}, \ldots$.

Definition 8.15. If r is any real number, and a is a positive real number, we define $a^r = \lim_{x \to r} a^x$ for x varying over the rational numbers. We say that a is the *base* and r is the *exponent.*

Remark 8.16. We can't actually raise a negative real number to an irrational power. The limit would vary over x with even denominator, and a^x is not defined if x has even denominator and $a < 0$.

Proposition 8.17. *The exponential function $f_a(x) = a^x$ is well-defined for any r when $a > 0$, and is continuous on all real numbers. Further, it satisfies the exponential laws:*

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$.

Proposition 8.18. *If $a > 1$, then $\lim_{x \to +\infty} a^x = +\infty$ and $\lim_{x \to -\infty} a^x = 0$.*

If $0 < a < 1$ then $\lim_{x \to +\infty} a^x = 0$ and $\lim_{x \to -\infty} a^x = +\infty$.

Proof. Both of these can be seen by considering cases where x is an integer. □

There is a number which we will see works much better as a base for the exponential function than any other. This is the number

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

It's possible to prove that this limit exists, but not incredibly easy. It happens that $e \approx 2.71828$. We often write exp for the exponential function with base e; that is, $\exp(x) = e^x$.

Remark 8.19. The number e was discovered by Jacob Bernoulli in the context of compound interest. If your interest rate is r and it's compounded n times a year, then the growth rate per year is $(1 + \frac{r}{n})^{1/n}$. If the interest is "compounded continuously," your money grows at a rate equal to the limit of this expression as n goes to $+\infty$—which is e. The number was named by Leonhard Euler (hence the "e") when he used it for logarithms.

We'd like to compute the derivative of exp, and also of a^x for a positive real number a. This is a bit difficult to do directly. So we will, as usual, cheat.

8.2.1 Logarithms

The exponential function $f(x) = a^x$ is one-to-one, since if $f(x) = f(y)$, then $a^x = a^y$, which means that $a^{x-y} = 1$ and so $x - y = 0$. So a^x must have an inverse function.

Definition 8.20. The *logarithmic function with base a*, written $\log_a$, is the inverse function to a^x. It has domain $(0, +\infty)$, and its image is all real numbers. We often write ln for $\log_e$.

Thus if $a > 0$, we see that $\log_a(a^x) = x$ for every real x, and $a^{\log_a(x)} = x$ for every $x > 0$.

Example 8.21.

- $\log_3(9) = 2$.
- $\log_2(8) = 3$

- $\log_a(1) = 0$ for any $a > 0$.

Proposition 8.22. *If $a > 1$, then $\lim_{x\to+\infty}\log_a(x) = +\infty$ and $\lim_{x\to-^+}\log_a(x) = -\infty$.*

The logarithm also has a number of properties corresponding to the exponential laws:

Proposition 8.23.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r\log_a(x)$ *for any real number r.*

Example 8.24.

- $\ln(a) + \frac{1}{2}\ln(b) = \ln(a) + \ln(b)^{1/2} = \ln(a\sqrt{b})$.
- Solve $e^{5-3s} = 10$. We have that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Remark 8.25. These properties are actually historically why the logarithm was originally important. Before calculators, people doing difficult computational work had to work by hand. Adding five digit numbers is much, much easier than multiplying them. So engineers would take the log of the numbers, add them together, and then exponentiate. This was all done with the help of massive books called log tables that would tell you the logarithm of a given number. Slide rules are essentially a way of making the log tables portable; but they were superseded by pocket calculators.

There is one more important logarithmic formula, corresponding to the exponential law I left out:

Proposition 8.26 (change of base)**.** *For any positive number $a \neq 1$, we have $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.*

Proof. $\exp(\log_a(x)\cdot\ln(a)) = a^{\log_a(x)} = x$, so $\log_a(x)\cdot\ln(a) = \ln(x)$. □

This allows us to convert logs in any base to logs in another base.

Example 8.27. What is $\log_2 10$? By the change of base formula, we have $\log_2(10) = \frac{\ln 10}{\ln 2}$. $\ln 10 \approx 2.3$ and $\ln 2 \approx .7$, so $\log_2 10 \approx 2.3/.7 \approx 23/7$.

8.3 Derivatives of exponentials and logs

Now we're ready to start computing derivatives. Recall that $e = \lim_{x\to 0}(1+x)^{1/x}$.

Proposition 8.28. *The function $f(x) = \log_a(x)$ is differentiable, with derivative $f'(x) = \frac{1}{x}\log_a e$.*

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a(x+h)-\log_a(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a((x+h)/x))}{h} \\
&= \lim_{h\to 0} \frac{\log_a(1+\frac{h}{x})}{h} \\
&= \frac{1}{x}\lim_{h\to 0} \frac{x}{h}\log_a(1+\frac{h}{x}) \\
&= \frac{1}{x}\lim_{h\to 0} \log_a\left((1+\frac{h}{x})^{x/h}\right) \\
&= \frac{1}{x}\log_a\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h/x}}\right) \\
&= \frac{1}{x}\log_a(e)
\end{aligned}$$

□

Corollary 8.29. *If* $f(x) = \log_a(x)$ *then* $f'(x) = \frac{1}{x \ln a}$.

Proof. By the change of base formula, $\log_a(e) = \frac{\ln(e)}{\ln(a)}$. □

Corollary 8.30. $\ln'(x) = \frac{1}{x}$.

Remark 8.31. An alternate path to discover the natural logarithm is to ask "what is the function whose derivative is $1/x$?" We will mention this line of thought briefly at the end of class.

Example 8.32.

- Let $f(x) = \ln(x^3+1)$. Then $f'(x) = \frac{1}{x^3+1} \cdot 3x^2$.
- Let $g(x) = \log_a(\cos(x))$. Then $g'(x) = \frac{1}{\cos(x)\ln(a)} \cdot (-\sin(x)) = -\tan(x)/\ln(a)$.
- If $h(x) = \ln(|x|)$ then $h'(x) = 1/x$ if $x > 0$ and $h'(x) = (-1/x)\cdot(-1) = 1/x$ if $x < 0$. So $h'(x) = \frac{1}{x}$.

We can sometimes use logarithms and implicit differentiation to make difficult differentiation problems easier, just as we use them to simplify difficult arithmetic problems.

Example 8.33 (Power Rule)**.** If r is a real number and $f(x) = x^r$, then

$$\begin{aligned} y &= x^r \\ \ln|y| &= r\ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \\ \frac{dy}{dx} &= r\frac{y}{x} = rx^{r-1}. \end{aligned}$$

And finally, we can use the logarithmic derivatives to figure out the derivative of exp.

Proposition 8.34. *If $f(x) = a^x$ for $a > 0$, then f is differentiable and $f'(x) = a^x \ln a$.*

Proof.

$$\begin{aligned} y &= a^x \\ \ln|y| &= x\ln|a| \\ \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y\ln a = a^x \ln a. \end{aligned}$$

□

Corollary 8.35. $\exp'(x) = \exp(x)$.

Example 8.36.

- If $f(x) = e^{\sin(x)}$ then $f'(x) = e^{\sin(x)} \cdot \cos(x)$.
- If $g(x) = 5^{x^2+1}$ then $g'(x) = \ln(5)5^{x^2+1} \cdot 2x$.

Poll Question 8.3.1. If $h(x) = x^x$ we have to be *very careful*—the obvious approaches don't actually work. But logarithmically:

$$\begin{aligned} y &= x^x \\ \ln|y| = x\ln|x| & \\ \frac{1}{y}\frac{dy}{dx} &= \ln|x| + \frac{x}{x} = \ln|x| + 1 \\ \frac{dy}{dx} &= x^x(\ln|x| + 1). \end{aligned}$$

So $h'(x) = (\ln|x| + 1)x^x$.

You can get the same result by writing $h(x) = e^{x\ln(x)}$, and thus $h'(x) = e^{x\ln(x)}(\ln(x)+1) = x^x(\ln(x) + 1)$.

Example 8.37. We wish to find the derivative of $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

$$\begin{aligned}
\ln y &= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4x} + \frac{2x}{2x^2+2} - \frac{3\cdot 5}{3x+2} \\
\frac{dy}{dx} &= y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \\
&= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).
\end{aligned}$$

8.4 Inverse Trigonometric Functions

We can invert some polynomials, and we can invert exponential functions. The other very common sort of function to work with is a trigonometric function, and we'd like to find inverses to these as well.

As a straightforward question, we cannot invert the trigonometric functions because they are all periodic, and thus not one-to-one. For instance, $\sin(0) = \sin(\pi) = \sin(2\pi) = \sin(n\pi)$ for any integer n.

However, sometimes a function is invertible if you restrict its domain enough, e.g. to between two critical points. In this section we make canonical domain choices for the trigonometric functions such that they are invertible.

Definition 8.38. If $-1 \leq x \leq 1$, we define $\arcsin(x) = \sin^{-1}(x) = y$ where $\sin(y) = x$ and $-\pi/2 \leq y \leq \pi/2$.

arcsin has a domain of $[-1, 1]$ and a range of $[-\pi/2, \pi/2]$.

Example 8.39. We can determine that $\arcsin(-\sqrt{3}/2) = -\pi/3$ since $\sin(-\pi/3) = -\sqrt{3}/2$. (Of course, $\sin(5\pi/3) = -\sqrt{3}/2$ as well, but $5\pi/3 > \pi/2$).

With more cleverness, we can calculate $\cos(\arcsin(1/3))$. Suppose $\theta = \arcsin(1/3)$. Then θ is the angle of a triangle with opposite side of lenght 1 and hypotenuse of length 3; using the Pythagorean theorem we determine that the other side has length $\sqrt{8} = 2\sqrt{2}$. Since $\cos(\theta)$ is the length of the adjacent side over the hypotenuse, we have $\cos(\arcsin(1/3)) = 2\sqrt{2}/3$.

We can make similar definitions for inverse cosine and inverse tangent functions. We do have to be careful about the precise domains and images.

Definition 8.40. If $-1 \leq x \leq 1$, we define $\arccos(x) = \cos^{-1}(x) = y$ where $\cos(y) = x$ and $0 \leq y \leq \pi$. This function has domain $[-1, 1]$ and range $[0, \pi]$.

If x is a real number, we define $\arctan(x) = \tan^{-1}(x) = y$ where $\tan(y) = x$ and $-\pi/2 < y < \pi/2$. This function has domain $(-\infty, +\infty)$ and image $(-\pi/2, \pi/2)$.

$\lim_{x\to+\infty} \arctan(x) = \pi/2$ and $\lim_{x\to-\infty} \arctan(x) = -\pi/2$.

sin and cos and tan are all differentiable functions, so by the Inverse Function Theorem, so are arcsin and arccos and arctan, at least most of the time.

Proposition 8.41.

- $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$
- $\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}$
- $\arctan'(x) = \frac{1}{1+x^2}$.

Proof. There are two approaches to proving these facts. One involves trigonometric identities, and the other involves thinking about triangles. They both involve implicit differentiation.

Suppose $y = \arcsin(x)$. Then $\sin(y) = x$ and thus $\cos(y)\frac{dy}{dx} = 1$. Then we have $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

From here, we can say two things. One is that $\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$, using the trigonometric identity that $\cos^2(y) + \sin^2(y) = 1$ and being careful about sign choices.

I find it easier to think the following thing: if $y = \arcsin(x)$ then y is the angle of a triangle where the opposite side has length x and the hypotenuse has length 1. Then the other side has length $\sqrt{1-x^2}$, so $\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Note we got the same answer both ways, and they both involved basically the same facts; the identity $\sin^2(y) + \cos^2(y) = 1$ holds precisely because of the triangle argument. Either way you want to think of it is fine with me.

We can do the same with $\arccos(x)$. $\cos(y) = x$, so $\frac{dy}{dx} = \frac{-1}{\sin(y)} = -\frac{1}{\sqrt{1-x^2}}$.

arctan is slightly trickier. $\tan(y) = x$ so $\sec^2(y)\frac{dy}{dx} = 1$, and thus we have $\frac{dy}{dx} = \cos^2(y)$. Again, we can use the identity $1 + \tan^2(y) = \sec^2(y)$, but if we don't remember that we can see that y is the angle of a triangle with opposite side x and adjacent side 1, and hence hypotenuse $\sqrt{1+x^2}$. Then $\cos(y) = \frac{1}{\sqrt{1+x^2}}$ and so $\arctan'(x) = \cos^2(y) = \frac{1}{1+x^2}$. □

Example 8.42. What is $\arcsin'(.75)$? $\frac{1}{\sqrt{1-9/16}} = \frac{1}{\sqrt{7/16}}$.

What is $\arctan'(e^x)$? $\frac{1}{1+e^{2x}} \cdot e^x$.

What is $\arccos'(x^2+2x+3)$? $\frac{1}{\sqrt{1-(x^2+2x+3)^2}} \cdot (2x+2)$.

8.5 L'Hôpital's Rule

We often find ourselves wanting to evaluate limits of "indeterminate form": that is, the limit of a quotient whose numerator and denominator both approach 0 or both approach $\pm\infty$. In the past we've used various tricks to work out such limits, but today we develop a new and widely-applicable tool. This tool is especially useful for dealing with limits involving ln or exp.

Theorem 8.43 (L'Hôpital's Rule)**.** *Suppose f and g are differentiable, and $g'(x) \neq 0$ near a, except possibly at a. Suppose either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. (In other words, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form). Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

Remark 8.44. Note that L'Hôpital's Rule *only* applies to limits of indeterminate form.

Proof. We won't prove this fully, but we will prove it in the case where $f(a) = g(a) = 0$, $g'(a) \neq 0$, and f' and g' are continuous at a.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{(f(x) - f(a))(x-a)}{(g(x) - g(a))(x-a)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Example 8.45.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} &= \lim_{x\to 3} \frac{2x - 4}{2x - 2} = \frac{2}{4} = \frac{1}{2}. \\
\lim_{x\to 0} \frac{1 - \cos(x)}{\sin(x)} &= \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 1} \frac{\ln x}{x - 1} &= \lim_{x\to 0} \frac{1/x}{1} = 1.
\end{aligned}$$

Sometimes we have to apply L'Hôpital's rule more than once to get the results we want.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2} \\
&= \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0} \frac{\tan x}{3x} \\
&= \lim_{x\to 0} \frac{\sec^2(x)}{3} = \frac{1}{3}.
\end{aligned}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

We can also use L'Hôpital's rule to evaluate limits at infinity.

Example 8.46.

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{x^2+5x+3}{x^2+7x-2} &= \lim_{x\to\pm\infty} \frac{2x+5}{2x+7} \\
&= \lim_{x\to\pm\infty} \frac{2}{2} = 1. \\
\lim_{x\to+\infty} \frac{\ln(x)}{x} &= \lim_{x\to+\infty} \frac{1/x}{1} = 0. \\
\lim_{x\to+\infty} \frac{e^x}{x} &= \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.
\end{aligned}$$

In fact, it's not too hard to see, using L'Hôpital's Rule, that $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$ and $\lim_{x\to+\infty} \frac{\ln(x)}{x^n} = 0$.

Remember that L'Hôpital's rule only applies if we start with an indeterminate form.

Example 8.47.

$$\begin{aligned}
\lim_{x\to\pi} \frac{\sin(x)}{1-\cos(x)} &\neq \frac{\cos(x)}{\sin(x)} = \pm\infty \\
\lim_{x\to\pi} \frac{\sin(x)}{1-\cos(x)} &= \frac{0}{1-(-1)} = 0.
\end{aligned}$$

A more dangerous example:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^3} = \lim_{x\to 0} \frac{e^x - 1}{3x^2} = \lim_{x\to 0} \frac{e^x}{6x}$$

You might think we should use L'Hôpital's rule again here; that would give $\lim_{x\to 0} \frac{e^x}{6} = 1/6$. But the top goes to 1 and the bottom goes to 0, so this is not an indeterminate form! The true limit is $\pm\infty$.

And sometimes L'Hôpital's rule doesn't alwasy work the way we'd like it to, just "because it doesn't."

Example 8.48.

$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\pm\infty} \frac{1}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to\pm\infty} \frac{\sqrt{x^2+1}}{x}$$

But here if we're clever we can observe that if the limit exists, then

$$\left(\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}\right)^2 = 1$$
$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \pm 1.$$

We can often use L'Hôpital's rule to compute limits of other indeterminate forms with a bit of cleverness. Recall the "minor" indeterminate forms are $1^\infty, \infty - \infty, 0^0, 0^\infty, 0 \cdot \infty$. Products can obviously be rewritten as quotients, and sums or differences can often be combined into something by collecting common denominators. Exponents can be turned into ratios by means of logarithms.

Example 8.49.

$$\begin{aligned}
\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) \\
&= \lim_{x\to\pi/2} \frac{1-\sin(x)}{\cos(x)} \\
&= \lim_{x\to\pi/2} \frac{-\cos(x)}{-\sin(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 0} \cot(2x)\sin(6x) &= \lim_{x\to 0} \frac{\sin(6x)\cos(2x)}{\sin(2x)} = 1 \cdot \lim_{x\to 0} \frac{\sin(6x)}{\sin(2x)} \\
&= \lim_{x\to 0} \frac{6\cos(6x)}{2\cos(2x)} = 3. \\
\lim_{x\to 1} x^{1/(1-x)} &= \exp\left(\lim_{x\to 1} \frac{\ln x}{1-x}\right) \\
&= \exp\left(\lim_{x\to 1} \frac{1/x}{-1}\right) = \lim_{x\to 1} e^{-1/x} = 1/e. \\
\lim_{x\to 0} (2^x + x)^{1/x} &= \exp\left(\lim_{x\to 0} \frac{\ln(2^x+x)}{x}\right) = \exp\left(\lim_{x\to 0} \frac{2^x \ln 2 + 1}{2^x + x}\right) \\
&= \exp\left(\lim_{x\to 0} \frac{2^x (\ln 2)^2}{2^x \ln(2) + 1}\right) \\
&= \exp\left(\lim_{x\to 0} \frac{2^x (\ln 2)^3}{2^x (\ln 2)^2}\right) \\
&= \exp(\ln 2) = 2.
\end{aligned}$$