Math 114 Spring 2018
Calculus I HW 2 Solutions
Due Friday, February 2

1. Let $f(x) = 2x + 1$, and let $L = 3$.

   (a) Suppose we have an error margin of $\epsilon = 1/10$, that is, we would like the distance between $f(x)$ and L to be less than $1/10$. What open interval does x need to be in to make this happen?

   (b) Now suppose our error margin is $\epsilon = 1/50$. Give an open interval for x so that the distance between $f(x)$ and L is less than $1/50$ for every x in the interval.

   Solution:

   (a) We have that $|f(x) - L| < \epsilon$, and thus $|2x - 2| < 1/10$, and thus $|x - 1| < 1/20$. This tells us that the distance from x to 1 is less than $1/20$, so x must be in the interval $(1 - 1/20, 1 + 1/20) = (.95, 1.05)$.

   Alternatively, we can write that $3 - \epsilon < f(x) < 3 + \epsilon$, and thus $2.9 < 2x + 1 < 3.1$. Solving this gives $1.9 < 2x < 2.1$ and thus $.95 < x < 1.05$.

   (b) We have that $|f(x) - L| < \epsilon$ and thus $|2x - 2| < 1/50$, and thus $|x - 1| < 1/100$. Thus x is in the interval $(1 - 1/100, 1 + 1/100) = (.99, 1.01)$.

   Alternatively, again, we can take $2.98 < 2x + 1 < 3.02$, so $1.98 < 2x < 2.02$, so $.99 < x < 1.01$.

2. Find, with proof, $\lim_{x\to 3} 4x$.

   Solution: We guess $4 \cdot 3 = 12$.

   Let $\epsilon > 0$ and let $\delta = \underline{\epsilon/4}$. Then if $|x - 3| < \delta$, we have

$$|4x - 12| = |4(x - 3)| = 4|x - 3| < 4\delta = 4\epsilon/4 = \epsilon.$$

3. Find, with proof, $\lim_{x\to 2}(x + 1)^2$.

   Solution: We guess $(2 + 1)^2 = 9$.

   Let $\epsilon > 0$ and let $\delta < \underline{\epsilon/7, 1}$. Then if $|x - 2| < \delta$ we have

$$\begin{aligned} |(x+1)^2 - 9| &= |x^2 + 2x + 1 - 9| = |x^2 + 2x - 8| = |(x^2 - 4) + 2(x - 2)| \\ &\le |x - 2| \cdot |x + 2| + 2|x - 2| \le |x + 2|\delta + 2\delta \\ &= \delta(2 + |x - 2 + 4|) \le \delta(2 + |x - 2| + 4) \le \delta(7) < \epsilon. \end{aligned}$$

Alternate Solutions: We can compute $|(x+1)^2 - 9| < \delta(6+\delta)$ and then solve the quadratic equation $\delta^2 + 6\delta - \epsilon = 0$ for δ, giving us $\delta = \frac{-6\pm\sqrt{36+4\epsilon}}{2}$. Thus if $x < \delta = \frac{-6+\sqrt{36+4\epsilon}}{2} = -3 + \sqrt{9+\epsilon}$ then $|f(x) - 9| < \epsilon$.

Alternatively again, we can observe

$$|(x+1)^2 - 9| = |x^2 + 2x - 8| = |x+4| \cdot |x-2| < |x+4|\delta \le \delta(|x-2| + 6)$$

which then follows through into either of the previous solutions.

4. Find, with proof, $\lim\limits_{x\to 1} x^2$.

Solution: We guess $1^2 = 1$.

Let $\epsilon > 0$ and set $\delta < \underline{\epsilon/3, 1}$. Then if $|x-1| < \delta$ we compute

$$\begin{aligned}|x^2 - 1| &= |x-1| \cdot |x+1| = |x-1| \cdot |x-1+2| \le |x-1|(|x-1| + 2) \\ &< \delta(1+2) < 3\epsilon/3 = \epsilon.\end{aligned}$$

Alternate Solutions: We can observe that $|x-1|(|x-1|+2) < \delta(\delta+2)$ and solve $\delta^2 + 2\delta - \epsilon = 0$ for δ, giving $\delta = \frac{-2\pm\sqrt{4+4\epsilon}}{2} = -1 \pm \sqrt{1+\epsilon}$. Then we observe that if $x < \delta = -1 + \sqrt{1+\epsilon}$ then $|x^2 - 1| < \epsilon$.

5. Find, with proof, $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x-3}$.

Solution: We guess that we can cancel out an $x-3$, and thus get $3+3 = 6$.

Let $\epsilon > 0$ and set $\delta = \underline{\epsilon}$. Then if $0 < |x+3| < \delta$, we compute

$$\begin{aligned}\left|\frac{x^2-9}{x-3} - 6\right| &= \left|\frac{(x-3)(x+3)}{x-3} - 6\right| = |(x+3) - 6| \\ &= |x-3| < \delta = \epsilon.\end{aligned}$$

6. $\star$ Find, with proof, $\lim\limits_{x\to 2} \dfrac{1}{x-1}$.

Solution: We guess $1/(2-1) = 1$.

Let $\epsilon > 0$ and set $\delta < \underline{\epsilon/2, 1/2}$. Then if $|x-2| < \delta$, we compute

$$\left|\frac{1}{x-1} - 1\right| = \left|\frac{1 - 1(x-1)}{x-1}\right| = \left|\frac{2-x}{x-1}\right| = \frac{|x-2|}{|x-1|}$$

But we know that

$$|x-1| = |x-2+1| = |1-(2-x)| \ge 1 - |x-2| \ge 1 - \delta \ge 1/2$$

and thus

$$\frac{|x-2|}{|x-1|} \le \frac{\delta}{1-\delta} < \frac{\epsilon/2}{1/2} = \epsilon.$$

7. ($\star$) Find (with proof) $\lim_{x\to 5} \frac{1}{x-4}$.

Solution: Let $\epsilon > 0$ and let $\delta \leq \underline{1/2, \epsilon/2}$. Then if $|x-5| < \delta$, we compute

$$\begin{aligned} \left|\frac{1}{x-4} - 1\right| &= \left|\frac{1-(x-4)}{x-4}\right| \\ &= \frac{|5-x|}{|x-4|} < \frac{\delta}{|x-4|}. \end{aligned}$$

Since the denominator is positive, we need to make the denominator big, and so use the reverse triangle inequality. Then we compute

$$|x-4| = |(x-5)+1| = |1-(5-x)| \geq 1 - |5-x| > 1 - \delta \geq 1/2$$

after we set $\delta \leq 1/2$. Thus

$$\left|\frac{1}{x-4} - 1\right| < \frac{\delta}{|x-4|} < \frac{\delta}{1/2} = 2\delta < \epsilon.$$