Math 114 Spring 2018
Calculus I HW 3 Solutions
Due Wednesday, September 20

1. Let
$$f(x) = \begin{cases} 1 & x < 2 \\ 2 & x = 2 \\ 3 & x > 2 \end{cases}$$
What is $f(2)$? Prove that $\lim_{x\to 2} f(x)$ does not exist.

Solution: $f(2) = 2$ by definition of the function.

Suppose $\lim_{x\to 2} f(x) = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x-2| < \delta$ then $|f(x) - L| < \epsilon = 1$.

Let $x_1 = 2+\delta/2$, so that $|x_1-2| = \delta/2 < \delta$; then $f(x_1) = 3$, and thus $1 > |f(x_1)-L| = |3 - L|$.

Let $x_2 = 2 - \delta/2$, so that $|x_2 - 2| = \delta/2 < \delta$; then $f(x_2) = 1$ and $1 > |f(x_2) - L| = |1 - L| = |L - 1|$. (We flip the sign inside the absolute value to make the next step easier, when we would like the Ls to cancel).

Adding these two inequalities and applying the triangle inequality gives us
$$1 + 1 > |L - 1| + |3 - L| \geq |(L - 1) + (3 - L)| = 2,$$
but this is a contradiction, so the limit must not exist.

2. Let
$$j(x) = \begin{cases} 3x - 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$$
Show that $\lim_{x\to 3} j(x) = 7$.

Solution: Let $\epsilon > 0$ and fix $\delta \leq \underline{3, \epsilon/2}$. Then if $0 < |x - 3| < \delta$ we know that $x > 0$, and so we have
$$|j(x) - 7| = |2x + 1 - 7| = |2x - 6| = 2|x - 3| < 2\delta \leq \epsilon.$$

3. ($\star$)For the same function j, show that $\lim_{x\to 0} j(x)$ does not exist.

Solution: Suppose $\lim_{x\to 0} j(x) = L$. Fix $\epsilon = \underline{1}$ and suppose $\delta > 0$. Pick $x_1 = \delta/2$ and $x_2 = -\delta/2$.

Then since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |2\delta/2 + 1 - L| = |\delta + 1 - L|.$$

Similarly, since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |-3\delta/2 - 1 - L| = |L + 1 + 3\delta/2|.$$

Adding these equations gives

$$\begin{aligned} 2\epsilon &> |\delta + 1 - L| + |L + 1 + 3\delta/2| \\ &\geq |\delta + 1 - L + L + 1 + 3\delta/2| = |5\delta/2 + 2| \\ &= 2 + 5\delta/2. \end{aligned}$$

Since $\epsilon = 1$ this gives us $2 > 2 + 5\delta/2$ and thus $0 > \delta$ which is a contradiction. So no such limit exists.

4. ($\star$) Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution: Suppose $\lim_{x\to 0} \frac{|x|}{x} = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x| < \delta$ then $\left|\frac{|x|}{x} - L\right| < \epsilon = 1$.

Let $x_1 = \delta/2 < \delta$; then

$$1 = \epsilon > \left|\frac{|x_1|}{x_1} - L\right| = \left|\frac{|\delta/2|}{\delta/2} - L\right| = |1 - L|.$$

Let $x_2 = -\delta/2$ so that $|x_2| = \delta/2 < \delta$; then

$$1 = \epsilon > \left|\frac{|x_2|}{x_2} - L\right| = \left|\frac{|-\delta/2|}{-\delta/2} - L\right| = |-1 - L| = |L + 1|.$$

Then adding the two inequalities and using the triangle inequality gives us

$$2 = 1 + 1 > |L + 1| + |1 - L| \geq |(L + 1) + (1 - L)| = 2,$$

but this is a contradiction, so the limit must not exist.

5. From the definition, prove that $\lim_{x\to 2} \frac{1}{x-2} = \pm\infty$.

Solution: Let $N > 0$ and set $\delta = \underline{1/N}$. Then if $0 < |x - 2| < \delta$, then

$$\left|\frac{1}{x-2}\right| = \frac{1}{|x-2|} > \frac{1}{\delta} = \frac{1}{1/N} = N.$$

Thus $\lim_{x\to 2} \frac{1}{x-2} = \pm\infty$.

6. From the definition, prove that $\lim_{x\to -1} \frac{4}{(x+1)^2} = +\infty$.

Solution: Let $N > 0$ and set $\delta = \underline{2/\sqrt{N}}$. Then if $0 < |x + 1| < \delta$, then

$$\frac{4}{(x+1)^2} > \frac{4}{\delta^2} \geq \frac{4}{(2/\sqrt{N})^2} = N.$$