Lab 6 Thursday March 22

Approximating Functions with Derivatives

Last lab, we drew secant lines, which are lines that intersect the graph of a function in (at least) two points; we may recall that by rearranging some information, we can write

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a).$$

As x approaches a, this becomes closer to being a tangent line, and the slope term becomes closer to $f'(a)$. Thus we can get a decent approximation, if x and a are close, by replacing this difference quotient with the derivative:

$$f(x) \approx f'(a)(x - a) + f(a).$$

Another approach to this same idea is to think geometrically, about the tangent line. Last week in lab we drew tangent lines, and saw that the tangent line to a function looks very similar to that function close up. That is, the tangent line to f at a is the line that most resembles f near the point a. Thus if we write the equation to the tangent line

$$y = f'(a)(x - a) + f(a)$$

then we get the line which most closely approximates the function f; we can plug x values into this formula to find approximate calculations of $f(x)$, with the approximation improving if x is closer to a.

If we want to compute `f[b]` for some specific and awkward value `b`, we can:

(a) Find a value a which is close to b, such that `f[a]` is easy to compute.

e.g. if $f(x) = \sqrt{x}$ and $b = 5$ we may pick $a = 4$.

(b) Calculuate the derivative at the point a: `f'[a]`.

If $f(x) = \sqrt{x}$ and $b = a$ then $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{4}$.

(c) Find an equation $y = m(x - a) + f(a)$ for the line tangent to f at a. (Check your work by plotting f and your line on the same plot in Mathematica). What is the y value given by this line when the x-coordinate is b?

We know the slope of the tangent line at a is $\frac{1}{4}$, and $f(a) = 2$, so our line is $y = \frac{1}{4}(x - 4) + 2$. We can plot this in Mathematica with
`Plot[{Sqrt[x], (1/4) (x-4) + 2},{x,2,6}]` or
`Plot[{Sqrt[x], Sqrt'[4](x-4) + Sqrt[4]},{x,2,6}]`
When we plug in 5 (our b value) for x we get $y = \frac{1}{4}(5 - 4) + 2 = \frac{1}{4} + 2 = \frac{9}{4}$. Thus we guess $\sqrt{5} \approx \frac{9}{4}$.

(d) Check your answer from (c) by evaluating `f[a]` (or more likely `N[f[a]]`) in Mathematica. How accurate were you?

Mathematica tells us that `N[Sqrt[5]]` is 2.23607, which is pretty close to 2.25. If we want to do better, in a few weeks we will cover a tool called "Newton's Method" that allows us to refine our answers and get closer approximations. Alternatively, the tool of "Taylor series" gives another improved method of approximations, which uses the both the first and second (and potentially higher) derivatives. We will talk about this in the last week of the course.

Exercises

Now answer the following questions. In each problem, before you do any computations, think carefully about what you should use for f, a, b.

1. Estimate $(2.1)^5$ without doing any calculations. Then use the tangent line to approximate $(2.1)^5$. How do these compare? Can you see your original estimate in your tangent line formula?

   Solution: The obvious rough estimate is $2^5 = 32$.

   We take $f(x) = x^5$ and $a = 2$. Then $f'(x) = 5x^4$, so we have $f(2) = 32, f'(2) = 80$, and

   $$f(2.1) \approx 80(2.1 - 2) + 32 = 40.$$

   The exact answer is 40.841. We can see our original estimate as the constant term in our tangent line approximation.

2. Now approximate $(2.5)^5$ using $a = 2$. Approximate 3^5 using $a = 2$. Are your approximations getting better or worse? What does this tell you about what counts as "close" to 2?

   Solution: We have

   $$\begin{aligned}(2.5)^5 &\approx 80 \cdot (2.5 - 2) + 32 = 72\\ 3^5 &\approx 80 \cdot (3 - 2) + 32 = 112.\end{aligned}$$

   The true answers are 97.6563 and 243. Unlike in part (a), these estimates are not especially good. This is because 3 is actually not very close to 2—especially proportionately. Of course, it's not that hard to compute 3^5 directly.

   These methods are best when $x - a$ is very small relative to everything else. We often use them in the real world for $x - a < .1$ or so.

3. Without calculating, find an upper bound and a lower bound for $(4.5)^3$. (Hint: $4 < 4.5 < 5$). Now approximate 4.5^3 with a tangent line in two different ways, from two different base points. What happens?

   Solution: For our bounds, we would expect $64 = 4^3 < 4.5^3 < 5^3 = 125$.

   We take $f(x) = x^3$ and $a = 4$. Then $f(a) = 64$, and $f'(x) = 3x^2$ so $f'(a) = 48$, and

   $$f(4.5) \approx 48(4.5 - 4) + 64 = 24 + 64 = 88.$$

   Alternatively, we can take $f(x) = x^3$ and $a = 5$. Then $f(a) = 125, f'(x) = 3x^2, f'(a) = 75$, and

   $$f(4.5) \approx 75(4.5 - 5) + 125 = 125 - 37.5 = 87.5.$$

   The exact answer is 91.125. These approximations are both decent but not great—as we'd expect, since 4.5 is close-ish to 4 and to 5, but not especially close.

4. Approximate `Sin[.05]` and `Cos[.05]` Note that this is .05 and not .5. Find a formula to approximate $\sin(x)$ and $\cos(x)$ when x is "small". (This is the revenge of the Small Angle Approximation).

   Solution: We take $a = 0$. Then since $\sin'(x) = \cos(x)$ and so $\sin'(0) = \cos(0) = 1$, we have

   $$\sin(.05) \approx 1(.05 - 0) + 0 = .05.$$

   This is basically the small angle approximation: for small x, we have $\sin(x) \approx x$. (The true answer is about .04998).

   Similarly, $\cos'(x) = -\sin(x)$ so $\cos'(0) = 0$. Then

   $$\cos(.05) \approx 0(.05 - 0) + 1 = 1.$$

   For small x, we have $\cos(x) = 1$. (The true answer is about .9986).

5. Now try `Sin[3/4]` and `Cos[3/4]` (Think about your choices for a here; you can do much better than 1) .

   Solution: Cheating only a little bit, we pick $a = \pi/4$, since $\pi \approx 3$. Then

   $$\sin(3/4) \approx \cos(\pi/4)(3/4 - \pi/4) + \sin(\pi/4) = \frac{\sqrt{2}}{8}(3 - \pi) + \frac{\sqrt{2}}{2} \approx \sqrt{2}/2(1 - .035)$$
   $$\cos(3/4) \approx -\sin(\pi/4)(3/4 - \pi/4) + \cos(\pi/4) = -\frac{\sqrt{2}}{8}(3 - \pi) + \frac{\sqrt{2}}{2} \approx \sqrt{2}/2(1 + .035).$$

6. Approximate `CubeRoot[28]` and `82^(1/4)`.

   Solution: We take $a = 27$ and $a = 81$ respectively.

   $$\sqrt[3]{28} \approx \frac{1}{3}(27)^{-2/3}(28 - 27) + 3 = \frac{1}{27} + 3 \approx 3.03704$$
   $$\sqrt[4]{82} \approx \frac{1}{4}(81)^{-3/4}(82 - 81) + 3 = \frac{1}{108} + 3 \approx 3.00926.$$

   The true answers are approximately 3.03659 and 3.00922 respectively.

7. Now approximate `28^3` and `82^4` from the same base points. Are these approximations better or worse than your approximations of `CubeRoot[28]` and `82^(1/4)` above? Why?

   Solution: We have

   $$28^3 \approx 3(27)^2(28 - 27) + 27^3 = 21870$$
   $$82^4 \approx 4(81)^3(82 - 81) + 82^4 = 45172485$$

   In contrast the true answers are 21952 and 45212176.

   These approximations aren't *terrible* but they aren't very good either. Since the derivative is changing quickly here (the second derivatives are $6 \cdot 27$ and $12 \cdot 81^2$ respectively), the approximation won't be very good.

8. Use the same method to find `CubeRoot[64.1]`.

   Solution: We take $a = 64, f(x) = \sqrt[3]{x}, f(a) = 4$. Then

$$\sqrt[3]{64.1} \approx \frac{1}{3}(64)^{-2/3}(64.1 - 64) + 4 = \frac{1}{48} + 4 \approx 4.02083.$$

   The true answer is about 4.00208.

9. Approximate $(1.01)^{10}$. Approximate $(1.01)^{\alpha}$ where $\alpha \neq 0$ is some constant (your answer will have an α in it).

   Now find a formula to approximate $(1+\epsilon)^{\alpha}$ where ϵ is a "small" constant and $\alpha \neq 0$ is a constant. (This rule is the called the "binomial approximation" and is often useful in physics).

   Solution: For the first part, our function is $f(x) = x^{10}$ and our $a = 1$. So $f(a) = 1$ and $f'(a) = 10a^9 = 10$. Then we have

$$f(1.01) \approx 10(1.01 - 1) + 1 = 1.1.$$

   The true answer is about 1.10462.

   For the second part, we have $f(x) = x^{\alpha}$, so $f'(x) = \alpha x^{\alpha-1}$. We again have $f(1) = 1$ and $f'(1) = \alpha(1)^{\alpha-1} = \alpha$, so

$$f(1.01) \approx \alpha(1.01 - 1) + 1 = 1 + \alpha/100.$$

   For the third part, we still take $f(x) = x^{\alpha}$ and $a = 1$. But we compute

$$f(1+\epsilon) \approx \alpha(1 + \epsilon - 1) + 1 = 1 + \alpha\epsilon.$$

   This formula is used constantly in physics and other applications.

10. If you take $a = 0$ and $f(x) = x^{10}$, use a tangent line to approximate $f(2)$. What happens and why? What if you instead approximate with $a = 1$?

    Solution: We have $f'(x) = 10x^9$, so we have $f'(0) = 0$, and thus

$$f(2) \approx 0(2 - 0) + 0 = 0.$$

    If we take $a = 1$, we have

$$f(2) \approx 10(2 - 1) + 1 = 11.$$

    The true answer is 1024, which is far away from both of those. In essence, the derivative is changing so quickly that the tangent line approximation is not very good over those distances.