Math 114 Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to -3} x^2$.

Solution: We guess 9.

Let $\epsilon > 0$ and let $\delta \leq \underline{1, \epsilon/7}$. Then if $0 < |x+3| < \delta$ then we have

$$\begin{aligned} |x^2 - 9| &= |x-3||x+3| < \delta|x-3| \\ &= \delta|x+3-6| \leq \delta(|x+3|+6) \\ &< \delta(\delta+6) < 7\delta \leq 7\epsilon/7 = \epsilon. \end{aligned}$$

(b) Directly from the definition, compute with proof $\lim_{x \to 2} \frac{x+4}{x}$.

Solution: We guess 3.

Let $\epsilon > 0$ and set $\delta \leq \underline{1, \epsilon/2}$. Then if $0 < |x-2| < \delta$ then

$$\left| \frac{x+4}{x} - 3 \right| = \left| \frac{x+4}{x} - \frac{3x}{x} \right| = \left| \frac{-2x+4}{x} \right| = \frac{2|x-2|}{|x|} < \frac{2\delta}{|x|}.$$

To handle the bottom, we observe $|x| = |x-2+2| \geq |2| - |x-2| > 2 - \delta \geq 1$. Thus we have

$$\left| \frac{x+4}{x} - 3 \right| < \frac{2\delta}{|x|} < \frac{2\delta}{1} = 2\delta \leq 2\epsilon/2 = \epsilon.$$

Problem 2.

Let

$$f(x) = \begin{cases} 5 & x < 2 \\ 1 & x > 2 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x \to 1} f(x)$.

Solution: We guess 5.

Let $\epsilon > 0$ and set $\delta = \underline{1}$. Then if $0 < |x-1| < \delta$, we see that $x - 1 < 1$ and thus $x < 2$, so $f(x) = 5$, and thus we have

$$|f(x) - 5| = |5 - 5| = 0 < \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x \to 2} f(x)$ does not exist.

Solution: Set $\epsilon = 2$ and suppose $\delta > 0$. Suppose $\lim_{x \to 2} f(x) = L$. Then set $x_1 = 2 + \delta/2, x_2 = 2 - \delta/2$, and we have

$$\begin{aligned} \epsilon &> |f(x_1) - L| = |f(2+\delta/2) - L| = |1 - L| \\ \epsilon &> |f(x_2) - L| = |f(2-\delta/2) - L| = |5 - L| \\ 2\epsilon &> |L-1| + |5-L| \geq |L - 1 + 5 - L| = |4| = 4 \end{aligned}$$

Since $\epsilon = 2$ this gives us $4 > 4$, which is a contradiction.

Problem 3.

Let

$$g(x) = \begin{cases} 3x+6 & x > 3 \\ 2x+5 & x < 3 \end{cases}$$

(a) Directly from the definition of a limit, compute with proof $\lim_{x\to -2} g(x)$

Solution: We guess 1.

Let $\epsilon > 0$ and let $\delta \leq \underline{5, \epsilon/2}$. Then if $0 < |x+2| < \delta$, we see that $x+2 < 5$ so $x < 3$ so $g(x) = 2x+5$. Then we have

$$|g(x) - 1| = |2x+5-1| = |2x+4| = 2|x+2| < 2\delta \leq 2\epsilon/2 = \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x\to 3} g(x)$ does not exist.

Solution:

Suppose $\lim_{x\to 3} g(x) = L$. Set $\epsilon = \underline{2}$ and let $\delta > 0$. Let $x_1 = 3 - \delta/2$ and $x_2 = 3 + \delta/2$. Then we have

$$\begin{aligned} \epsilon &> |g(x_1) - L| = |2(3-\delta/2)+5-L| = |11-\delta-L| = |L+\delta-11| \\ \epsilon &> |g(x_2) - L| = |3(3+\delta/2)+6-L| = |15+3\delta/2-L| \\ 2\epsilon &> |L+\delta-11| + |15+3\delta/2-L| \geq |4+5\delta/2| = 44+5\delta/2 > 4. \end{aligned}$$

Since $\epsilon = 2$ this gives us $4 > 4$, which impossible. So no such limit exists.

Problem 4. (a) Directly from the definition, prove that $\lim_{x\to 3} \frac{3}{x-3} = \pm\infty$.

Solution:

Let $N > 0$ and set $\delta = \underline{3/N}$. Then if $0 < |x-3| < \delta$ we have

$$\left|\frac{3}{x-3}\right| = \frac{3}{|x-3|} > \frac{3}{\delta} = \frac{3}{3/N} = N.$$

(b) Directly from the definition, prove that $\lim_{x\to 5} \frac{x}{(x-5)^2} = +\infty$.

Solution: Let $N > 0$ and set $\delta \leq \underline{1, 2/\sqrt{N}}$. Then if $|x-5| < \delta$ we have that $5 - x < 1$ and thus $x > 4$. Then we have

$$\begin{aligned} \frac{x}{(x-7)^2} &> \frac{4}{(x-7)^2} > \frac{4}{\delta^2} \\ &> \frac{4}{(2/\sqrt{N})^2} = N. \end{aligned}$$

Problem 5. Compute the following limits, showing each step and naming each limit law you use.

(a)

$$\lim_{x\to 5} x\sqrt{x+4} + 1$$

Solution:

$$\begin{aligned} \lim_{x\to 5} x\sqrt{x+4} + 1 &= \lim_{x\to 5} x\sqrt{x+4} + \lim_{x\to 5} 1 && \text{Additivity} \\ &= \lim_{x\to 5} x \lim_{x\to 5} \sqrt{x+4} + \lim_{x\to 5} 1 && \text{Products} \\ &= \lim_{x\to 5} x \sqrt{\lim_{x\to 5} x+4} + \lim_{x\to 5} 1 && \text{Exponents} \\ &= \lim_{x\to 5} x \sqrt{\lim_{x\to 5} x + \lim_{x\to 5} 4} + \lim_{x\to 5} 1 && \text{Additivity} \\ &= 5\sqrt{5 + \lim_{x\to 5} 4} + \lim_{x\to 5} 1 && \text{Identity} \\ &= 5\sqrt{5+4} + 1 = 16 && \text{Constants} \end{aligned}$$

(b)

$$\lim_{x \to -2} \frac{2x^2 + 6x + 4}{x + 2}$$

Solution:

$$\begin{aligned}
\lim_{x \to -2} \frac{2x^2 + 6x + 4}{x + 2} &= \lim_{x \to -2} 2\frac{(x+2)(x+1)}{x+2} && \text{Algebra} \\
&= \lim_{x \to -2} 2(x+1) && \text{Almost Identical Functions} \\
&= 2 \lim_{x \to -2} (x+1) && \text{Scalars} \\
&= 2\left(\lim_{x \to -2} x + \lim_{x \to -2} 1\right) && \text{Additivity} \\
&= 2\left(-2 + \lim_{x \to -2} 1\right) && \text{Identity} \\
&= 2\,(-2+1) = -2 && \text{Constants}
\end{aligned}$$