Math 114 Practice Test 2 Solutions

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Problem 1.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x \to 3} \frac{3-x}{\sqrt{x+1}-2} =$$

Solution:

$$\lim_{x \to 3} \frac{3-x}{\sqrt{x+1}-2} = \lim_{x \to 3} \frac{(3-x)(\sqrt{x+1}+2)}{x+1-4} = \lim_{x \to 3} -\sqrt{x+1} - 2 = -4.$$

(b)

$$\lim_{x \to 0} \frac{\sin(3x)\sin(2x)}{x \sin(4x)} =$$

Solution:

$$\lim_{x \to 0} \frac{\sin(3x)\sin(2x)}{x \sin(4x)} = \lim_{x \to 0} \frac{\sin(3x)}{3x} \frac{\sin(2x)}{2x} \frac{4x}{\sin(4x)} \frac{6}{4} = \frac{3}{2}.$$

by the small angle approximation.

(c)

$$\lim_{x \to +\infty} \frac{x^2+3x-1}{\sqrt{9x^4+4}} =$$

Solution:

$$\lim_{x \to +\infty} \frac{x^2+3x-1}{\sqrt{9x^4+4}} = \lim_{x \to +\infty} \frac{1+3/x-1/x^2}{\sqrt{9+4/x^4}} = 1/3.$$

(d)

$$\lim_{x \to -2} \frac{x^2+4x+4}{x^3-8} =$$

Solution:

$$\lim_{x \to -2} \frac{x^2+4x+4}{x^3-8} = \frac{0}{-16} = 0.$$

Problem 2.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 3^-} f(x) \qquad f(x) = \begin{cases} x-5 & x<3 \\ x^2+7x>3 \end{cases}$$

Solution:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x-5 = -2.$$

(b)

$$\lim_{x\to 1} \frac{x^2-3}{(x-1)^2} =$$

Solution: We know that $\lim_{x\to 1} x^2-3 = -2$ and $\lim_{x\to 1}(x-1)^2 = 0$, so the limit is $\pm\infty$. In fact, the bottom is always positive, and the top is negative, so the limit is $-\infty$.

(c)

$$\lim_{x\to 1} \frac{x^2-5x+4}{x^2+6x-7} =$$

Solution:

$$\lim_{x\to 1} \frac{x^2-5x+4}{x^2+6x-7} = \lim_{x\to 1} \frac{x-4}{x+7} = -3/8.$$

(d)

$$\lim_{x\to -1} \frac{x+4}{x+1} =$$

Solution: We see the limit of the top is 3 and the limit of the bottom is 0, so the limit is $\pm\infty$. The bottom can be positive or negative, so we can't say anything more than that.

Problem 3. (a) Using the Squeeze Theorem, show that

$$\lim_{x\to 1}(x-1)^2 \sin\left(\frac{1}{x-1}\right)$$

Solution: We know that $-1 \le \sin\left(\frac{1}{x-1}\right) \le 1$, and thus we have

$$-(x-1)^2 \le \sin\left(\frac{1}{x-1}\right) \le (x-1)^2.$$

We can compute (by continuity) that $\lim_{x\to 1}(x-1)^2 = 0$ and $\lim_{x\to 1} -(x-1)^2 = 0$. Thus by the squeeze theorem, $\lim_{x\to 1}(x-1)^2 \sin\left(\frac{1}{x-1}\right) = 0$.

(b) Let

$$f(x) = \begin{cases} x^2-1 & x>2 \\ 5-x & x<2 \end{cases}$$

If possible, define an extension of f that is continuous at all real numbers. **Solution:** f is defined and continuous at every point except 2. We see that

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2-1 = 3$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 5-x = 3$$
$$\lim_{x\to 2} f(x) = 3.$$

Thus we can define

$$f_{\text{fixed}}(x) = \begin{cases} x^2 - 1 & x \geq 2 \\ 5 - x & x \leq 2 \end{cases} = \begin{cases} x^2 - 1 & x > 2 \\ 3 & x = 2 \\ 5 - x & x < 2 \end{cases}$$

to be an extension of f that is continuous at all real numbers.

Problem 4. (a) Show that the function $f(x) = x^3 - 5x^2 + \sin(x) + 1$ has a real root.

Solution: We compute that $f(0) = 1$ and $f(-1) = -5 + \sin(-1) < -4$. Since f is continuous on the closed interval $[0, 1]$, by the intermediate value theorem we know that there is a c in $(0, 1)$ such that $f(c) = 0$.

(b) Directly from the definition of derivative, compute $f'(x)$ if $f(x) = \frac{3}{x}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \frac{3x - 3(x+h)}{x(x+h)} \\ &= \lim_{h\to 0} \frac{-3h}{xh(x+h)} = \lim_{h\to 0} \frac{-3}{x(x+h)} = \frac{-3}{x^2}. \end{aligned}$$

Problem 5. Compute the following derivatives using only the definition of derivative.

(a) Derivative of $f(x) = \sqrt{2x+7}$ at $x = 1$.

Solution:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+2h} - 3}{h} \\ &= \lim_{h\to 0} \frac{9 + 2h - 9}{h(\sqrt{9+2h} + 3)} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{9+2h} + 3} = \frac{1}{3}. \end{aligned}$$

(b) Derivative of $g(x) = -2x^2$ at $x = -1$.

Solution:

$$\begin{aligned} g'(-1) &= \lim_{h\to 0} \frac{g(-1+h) - g(-1)}{h} \\ &= \lim_{h\to 0} \frac{-2(-1+h)^2 + 2}{h} \\ &= \lim_{h\to 0} \frac{-2h^2 + 4h - 2 + 2}{h} \\ &= \lim_{h\to 0} \frac{4h - 2h^2}{h} \\ &= \lim_{h\to 0} 4 - 2h = 4. \end{aligned}$$