Math 114 Test 3 Solutions

Instructor: Jay Daigle

Problem 1. You may use any methods we have learned in class to solve these problems, but show enough work to justify your answers.

(a) If $f(x) = \dfrac{\tan(2x)}{2}$, find $f'(\pi/12)$.

Solution:

$$\begin{aligned} f'(x) &= \sec^2(2x) \\ f'(\pi/8) &= \sec^2(\pi/6) = (2/\sqrt{3})^2 = 4/3. \end{aligned}$$

(b) Find $\frac{d^2g}{dx^2}$ if $g(x) = \frac{\sin x}{x}$.

Solution:

$$\begin{aligned} g'(x) &= \frac{x\cos x - \sin x}{x^2} \\ g''(x) &= \frac{(\cos x - x\sin x - \cos x)x^2 - 2x(x\cos x - \sin x)}{x^4}. \end{aligned}$$

(c) Find an equation of the line tangent to $y = x\tan x$ at the point $x = \pi/4$.

Solution: $y' = \tan x + x\sec^2 x$ so $y'(\pi/4) = 1 + \pi/2$, and thus the equation for the tangent line is $y - \pi/4 = (1 + \pi/2)(x - \pi/4)$.

Problem 2. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \cos\left(\tan^3(x^2 + x)\right)$

Solution:

$$f'(x) = -\sin\left(\tan^3(x^2+2)\right) + 3\tan^2(x^2+x)\sec^2(x^2+x)(2x+1).$$

(b) $g(x) = \sec\left(\dfrac{x + \sqrt{x^2+1}}{\sin(x)}\right)$

Solution:

$$g'(x) = \sec\left(\frac{x+\sqrt{x^2+1}}{\sin(x)}\right)\tan\left(\frac{x+\sqrt{x^2+1}}{\sin(x)}\right)\frac{(1+\frac{1}{2}(x^2+1)^{-1/2}(2x))\sin(x) + \cos(x)(x+\sqrt{x^2+1})}{\sin^2 x}$$

Problem 3. (a) Find a tangent line to the curve given by $xy^2 = x + 9$ at the point $(3, -2)$.

Solution: We have

$$\begin{aligned} y^2 + 2xyy' &= 1 \\ y' &= \frac{1-y^2}{2xy} \\ &= \frac{1-4}{-12} = \frac{1}{4}. \end{aligned}$$

Alternatively

$$\begin{aligned} y^2 + 2xyy' &= 1 \\ 4 - 12y' &= 1 \\ 3 &= 12y' \\ y' &= 1/4. \end{aligned}$$

Thus the equation of the tangent line is

$$y + 2 = 1/4(x - 3).$$

(b) Find a formula for y' in terms of x and y if $\sec(x - 2y) = 5$.

Solution:

$$\begin{aligned} \sec(x-2y)\tan(x-2y)(1-2y') &= 0 \\ \sec(x-2y)\tan(x-2y) &= \sec(x-2y)\tan(x-2y)2y' \\ \frac{1}{2} &= y'. \end{aligned}$$

(c) Find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y if $y^4 = x^3 + y$.

Solution:

$$\begin{aligned} 4y^3y' &= 3x^2 + y' \\ (4y^3 - 1)y' &= 3x^2 \\ y' &= \frac{3x^2}{4y^3 - 1} \\ y'' &= \frac{6x(4y^3-1) - (12y^2y')3x^2}{(4y^3-1)^2} \\ &= \frac{6x(4y^3-1) - 12y^2\frac{3x^2}{4y^3-1}3x^2}{(4y^3-1)^2}. \end{aligned}$$

Problem 4. (a) Let $f(x) = (1 + x)^4$. Use a linear approximation of f to approximate 1.1^4.

Solution:

$$\begin{aligned} &f'(x) = 4(1+x)^3 \\ &f'(0) = 4 \\ &f(x) \approx f(0) + f'(0)(x-0) = 1 + 4x \\ &f(.1) \approx f(0) + f'(0)(.1 - 0) = 1 + 4(.1) = 1.4. \end{aligned}$$

(b) Suppose we have the differential equation $f'(x) = 2f(x) + x$, and $f(0) = 1$. Use Euler's method with two steps to estimate $f(1)$.

Solution:

$$\begin{aligned} f(1/2) &\approx f(0) + f'(0)(1/2 - 0) = 1 + (2 \cdot 1 + 0)(1/2) = 2 \\ f(1) &\approx f(1/2) + f'(1/2)(1 - 1/2) \approx 2 + (4 + 1/2)(1/2) = 2 + 9/4 = 17/4. \end{aligned}$$

Problem 5. (a) An oil spill has left a circular puddle of oil of uniform thickness, consisting of 4π cubic meters of oil. If the oil is spreading such that the radius increases by $1/10$ meters per hour, how quickly is the thickness decreasing when the radius is 20 meters?

(Hint: the formula for the volume of a cylinder is $V = \pi r^2 h$).

Solution: Let r be the radius and h be the thickness. Then we have $r = 20, V = 4\pi$, and $r' = 1/10$. We calculate

$$0 = \pi(2rr'h + r^2h')0 \qquad = 2r'h + rh'$$
$$rh' = -2r'h$$
$$20h' = -2(1/10)h$$
$$h' = \frac{-h}{100}$$

so we just need to calculate h. Plugging into the original equation gives

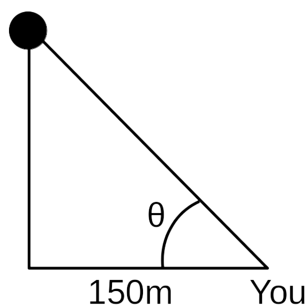
$$4\pi = \pi(20)^2h$$
$$h = \frac{4}{20^2} = \frac{1}{100}.$$

(This seems really small, but remember the units are meters—so this puddle is a centimeter high, which is actually pretty reasonable).

Thus we have

$$h' = \frac{-h}{100} = \frac{-1}{10000}.$$

(b) You are watching a New Year's Eve ball drop from a distance (along the ground) of 150 meters. If the ball falls at 20 meters per second, how quickly is the angle between your line of sight and the ground changing when it is ten seconds before the ball hits the ground at midnight? (Assume your eyes are at ground level).



Solution: At ten seconds before midnight, the ball has a height of 200 meters. We have

$$\tan\theta = \frac{h}{150}$$
$$\sec^2\theta \cdot \theta' = \frac{h'}{150} = \frac{-20}{150} = \frac{-2}{15}$$

But the hypotenuse of the triangle has length $\sqrt{200^2 + 150^2} = 250$, and we know that $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{250}{150} = \frac{5}{3}$. Thus we have

$$\frac{25}{9}\theta' = \frac{-2}{15}$$
$$\theta' = \frac{-18}{15 \cdot 25} = \frac{-6}{125}.$$