Math 212 Test 1 Solutions

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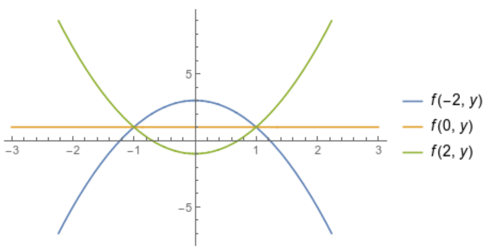
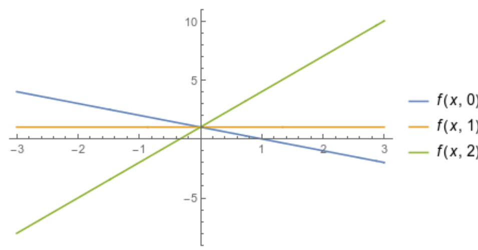
Problem 1. Let $f(x, y) = xy^2 - x + 1$

(a) Sketch cross-sections of f for $x = -2, 0, 2$ and $y = 0, 1, 2$.

Solution:





(b) If $\vec{u} = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$, compute $f_{\vec{u}}(1, 1)$.

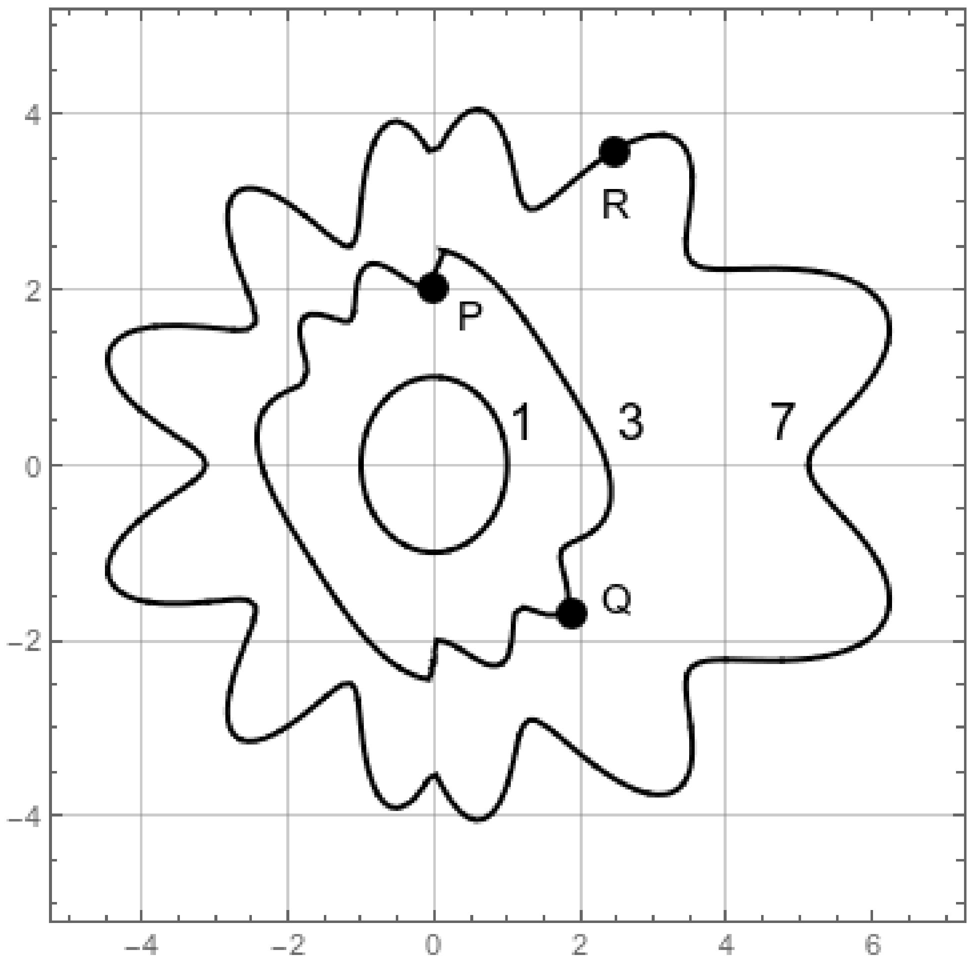
Solution: $\nabla f(x, y) = (y^2 - 1)\vec{i} + 2xy\vec{j}$, so

$$\nabla f(1, 1) = 2\vec{j}$$
$$f_{\vec{u}}(1, 1) = -6/5.$$

(c) At the point $(1, 2)$, in what direction should you move to increase $f(x, y)$ at the fastest possible rate? What is the rate of increase in that direction?

Solution: The direction of greatest increase is $\nabla f(1, 2) = 3\vec{i} + 4\vec{j}$. The rate of increase is $\|\nabla f(1, 2)\| = \sqrt{3^2 + 4^2} = 5$.

Problem 2. Here is a contour graph for a function $g(x, y)$.



(a) Estimate $\frac{\partial g}{\partial y}$ at the point P.

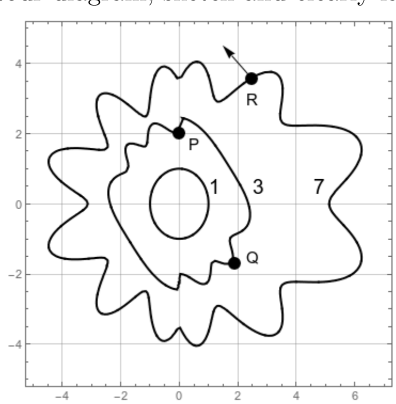
Solution: It should be a little bit less than 2. We see that going from $y = 1$ to $y = 2$ the value increases by 2, and going from $y = 2$ to $y = 4$ the value increases a little bit more than 4. So the estimate is a bit over 2, maybe 2.2 or so.

(b) Is the sign of the directional derivative of g at the point Q in the direction $\vec{i} - \vec{j}$ positive, negative, or zero?

Solution: Positive, since we're going from the $g = 3$ contour towards the $g = 7$ contour.

(c) On the contour diagram, sketch and clearly label an arrow in the direction of the gradient at R.

Solution:



Problem 3. (a) Give an equation for a plane through the points $(3,4,5),(1,4,1),(3,2,7)$.

Solution: There are two approaches here.

First, we can observe that the first two points share a y coordinate and the first and third share an x coordinate. Thus we can compute the x slope is 2 and the y slope is (-1). Then our equation is

$$z = 2(x-3) - (y-4) + 5.$$

Alternatively, we get the vectors $-2\vec{i} - 4\vec{k}$ and $2\vec{j} + 2\vec{k}$. Then we compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -4 \\ 0 & -2 & 2 \end{vmatrix} = -8\vec{i} + 4\vec{j} + 4\vec{k}\vec{n}$$

and thus the equation for the plane is

$$0 = -8(x-3) + 4(y-4) + 4(z-5).$$

These are, non-obviously, the same plane.

(b) Compute $(2\vec{i} + \vec{j} + 4\vec{k}) \times (-3\vec{i} - 3\vec{j} + 3\vec{k})$.

Solution: We compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 4 \\ -3 & -3 & 3 \end{vmatrix} = 15\vec{i} - 18\vec{j} - 3\vec{k}.$$

(c) Find an equation for the plane perpendicular to $3\vec{i} + 2\vec{j} - \vec{k}$ and $5\vec{i} + 7\vec{j}$ through the point $(5,2,2)$.

Solution: We compute the cross product

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 5 & 7 & 0 \end{vmatrix} = 7\vec{i} - 5\vec{j} + 11\vec{k}$$

So the tangent plane has equation

$$7(x-5) - 5(y-2) + 11(z-2) = 0.$$

Problem 4. (a) Compute $\nabla\sqrt{x+y^2+z^3}$.

Solution:

$$\frac{1}{2\sqrt{x+y^2+z^3}}\vec{i} + \frac{2y}{2\sqrt{x+y^2+z^3}}\vec{j} + \frac{3z^2}{2\sqrt{x+y^2+z^3}}\vec{k}.$$

(b) Compute $f_{\vec{v}}(-1,2)$ when $f(x,y) = xy^2 + x^3y$ and $\vec{v} = 2\vec{i} - \vec{j}$.

Solution:

$$\begin{aligned} \nabla f &= (y^2 + 3x^2y)\vec{i} + (2xy + x^3)\vec{j} \\ \nabla f(-1,2) &= 10\vec{i} - 5\vec{j} \\ f_{\vec{v}}(-1,2) &= (20+5)/\sqrt{5} = 5\sqrt{5}. \end{aligned}$$

(c) Suppose z is a function of x and y, and that we have $x = 3u - 2v$ and $y = 5u + v$. Find formulas for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= 3\frac{\partial z}{\partial x} + 5\frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= -2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}. \end{aligned}$$

Problem 5. (a) Find an equation for the tangent plane to the graph of the function $f(x, y) = 5x^2 + \sin(xy)$ at the point $(2, \pi/6)$.

Solution: We have $\nabla f(x, y) = (10x + y\cos(xy))\vec{i} + x\cos(xy)\vec{j}$, so $\nabla f(2, \pi/6) = (20 + \pi/6 \cdot 1/2)\vec{i} + \vec{j}$. Further we have $f(2, \pi/6) = 20 + \sqrt{3}/2$. Thus we get the equation

$$z = (20 + \pi/12)(x - 2) + (y - \pi/6) + 20 + \sqrt{3}/2.$$

(b) Let $g(x, y, z) = x^2y + y^2z$. Use a linear approximation at the point $(1, 2, 3)$ to estimate $g(.9, 2.1, 3.2)$.

Solution:

$$\begin{aligned} \nabla g(x, y, z) &= 2xy\vec{i} + (x^2 + 2yz)\vec{j} + y^2\vec{k} \\ \nabla g(1, 2, 3) &= 4\vec{i} + 13\vec{j} + 4\vec{k} \\ g(x, y, z) &\approx 4(x - 1) + 13(y - 2) + 4(z - 3) + 14 \\ g(.9, 2.1, 3.2) &\approx 4(-.1) + 13(.1) + 4(.2) + 14 = -.4 + 1.3 + .8 + 14 = 15.7. \end{aligned}$$