Math 212 Test 2 Solutions

Instructor: Jay Daigle

Problem 1. Let $f(x,y) = e^{xy-4} + x^2 y$.

(a) Find the degree 2 Taylor polynomial for f centered at $(2,2)$.

Solution:

We compute

$$\begin{aligned}
f_x(x,y) &= ye^{xy-4} + 2xy & f_x(2,2) &= 10\\
f_y(x,y) &= xe^{xy-4} + x^2 & f_y(2,2) &= 6\\
f_{xx}(x,y) &= y^2 e^{xy-4} + 2y & f_{xx}(2,2) &= 8\\
f_{xy}(x,y) &= e^{xy-4} + xye^{xy-4} + 2x & f_{xy}(2,2) &= 9\\
f_{yy}(x,y) &= x^2 e^{xy-4} & f_{yy}(x,y) &= 4.
\end{aligned}$$

Thus the Taylor polynomial is

$$T_2(x,y) = 9 + 10(x-2) + 6(y-2) + 4(x-2)^2 + 9(x-2)(y-2) + 2(y-2)^2.$$

(b) Use your answer in part (a) to estimate $f(1.9, 2.2)$.

Solution: We have

$$\begin{aligned}
f(1.9, 2.2) &\approx 9 + 10(-.1) + 6(.2) + 4(-.1)^2 + 9(-.1)(.2) + 2(.2)^2\\
&= 9 - 1 + 1.2 + .04 - .18 + .08 = 9.14.
\end{aligned}$$

Problem 2. (a) Find and classify the critical points of $f(x,y) = 2x^3 + 6xy + 3y^2$.

Solution: We have

$$\begin{aligned}
f_x(x,y) &= 6x^2 + 6y\\
f_y(x,y) &= 6x + 6y
\end{aligned}$$

This gives us $y = -x$, and thus we have $x^2 - x = 0$ so x is either 0 or 1. Thus our critical points are $(0,0)$ and $(1,-1)$.

We have

$$\begin{aligned}
f_{xx}(x,y) &= 12x & f_{xx}(0,0) &= 0 & f_{xx}(1,-1) &= 12\\
f_{xy}(x,y) &= 6 & f_{xy}(0,0) &= 6 & f_{xy}(1,-1) &= 6\\
f_{yy}(x,y) &= 6 & f_{yy}(0,0) &= 6 & f_{yy}(1,-1) &= 6.
\end{aligned}$$

Then for $(0,0)$ we have $D = 0 \cdot 6 - 6^2 = -36 < 0$, so we have a saddle point.

For $(1,1)$ we have $D = 12 \cdot 6 - 6^2 = 36 > 0$, and $f_{xx}(1,-1) = 12 > 0$. So this is a local minimum.

(b) Find the critical points of $g(x,y,z) = 9x - 6x^2 + x^3 + x^2 yz$.

Solution: We have

$$g_x(x,y,z) = 9 - 12x + 3x^2 + 2xyz$$
$$g_y(x,y,z) = x^2 z$$
$$g_z(x,y,z) = x^2 y$$

The third equation gives $x = 0$ or $y = 0$, and the second gives $x = 0$ or $z = 0$. If $x = 0$, then the first equation gives $1 = 0$ which is a contradiction, so we have $y = 0, z = 0$, and we solve $3x^2 - 12x + 9 = 0$.

Factoring this gives $0 = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$ so we have $x = 1$ or $x = 3$. Thus the two critical points are $(3, 0, 0)$ and $(1, 0, 0)$.

Problem 3. (a) Find the global extrema of $f(x,y) = 3xy - y + 5$ on the domain $x^2 + y^2 \leq 1$. Prove that they are global extrema.

Solution: The domain is closed and bounded, so by the Extreme Value Theorem there is a global maximum and a global minimum on the domain.

First we need interior critical points. We get

$$f_x(x,y) = 3y$$
$$f_y(x,y) = 3x - 1$$

and thus the only critical point is at $(1/3, 0)$. This is in the interior of the region. $f(1/3, 0) = 5$.

Now we check the boundary. We want to solve $3y = \lambda 2x$ and $3x - 1 = \lambda 2y$. The first equation gives $\lambda = \frac{3y}{2x}$. Plugging this into the second equation gives

$$3x - 1 = \frac{3y}{2x} 2y$$
$$6x^2 - 2x = 6y^2$$

but we know that $6y^2 = 6 - 6x^2$ so we get

$$6x^2 - 2x = 6 - 6x^2$$
$$12x^2 - 2x - 6 = 0$$

This gives us $x = \frac{1 \pm \sqrt{73}}{12} \approx -.63, .80$.

Then we have $y = \pm\sqrt{1 - x^2}$, so our four choices are the points $(-.63, -.78), (-.63, .78), (.80, -.61), (.80, .61)$. Plugging these values back into f gives us $7.24, 2.76, 4.16, 5.84$. Thus the global maximum is 7.24 which occurs at $(-.63, -.78)$, and the global minimum is 2.76 which occurs at $(.80, -.61)$.

(Note: I intended this problem to be numerically much simpler than it turned out to be, but I screwed up. Apologies! Hopefully we won't have a problem like this on any tests in the future. And I'll grade it with that awareness).

(b) Find the minimum value of $g(x,y,z) = x^2 + yz$ subject to the constraint $x + y + z = 5$. Justify the claim that this is a minimum. (This doesn't need to be a rigorous proof).

Solution: We have the equations $2x = \lambda, z = \lambda, y = \lambda$, and thus we have $z = y = 2x$. Plugging back into the constraint gives $x + 2x + 2x = 5$ so $x = 1$ and $y = z = 2$.

Thus the minimum value of g subject to this constraint is $1^2 + 2 \cdot 2 = 5$.

Problem 4. (a) Based on the following table of values for the function f, give an overestimate and an underestimate of $\int_R f \, da$ where R is the rectangle $0 \leq x \leq 4, 0 \leq y \leq 4$.

$y \setminus x$	0	2	4
0	1	3	7
2	2	5	9
4	4	6	10

Solution: For an underestimate, we have

$$1 \cdot 2 \cdot 2 + 3 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 + 5 \cdot 2 \cdot 2 = 4 + 12 + 8 + 20 = 44.$$

For an overestimate, we have

$$5 \cdot 2 \cdot 2 + 9 \cdot 2 \cdot 2 + 6 \cdot 2 \cdot 2 + 10 \cdot 2 \cdot 2 = 20 + 36 + 24 + 40 = 120.$$

(b) Sketch the region of integration, and compute $\int_0^3 \int_{\sqrt{y/3}}^1 \cos(x^3 - 1)\, dx\, dy$.

Solution:

$$\begin{aligned}\int_0^3 \int_{\sqrt{y/3}}^1 \cos(x^3 - 1)\, dx\, dy &= \int_0^1 \int_0^{3x^2} \cos(x^3 - 1)\, dy\, dx \\ &= \int_0^1 3x^2 \cos(x^3 - 1)\, dy \\ &= \sin(x^3 - 1)|_0^1 = \sin(0) - \sin(-1) = \sin(1).\end{aligned}$$

Problem 5. (a) Sketch the region R bounded by $x = 0, y = 2$, and $x + y = 4$, and compute $\int_R f\, dz$ for $f(x, y) = xy$.

Solution: We set u

$$\begin{aligned}\int_0^2 \int_2^{4-x} xy\, dy\, dx = \int_0^2 xy^2/2_2^{4-x}\, dx = \int_0^2 x(4-x)^2/2 - 2x\, dx - \int_0^2 6x - 4x^2 + x^3/2\, dx \\ = 3x^2 - 4x^3/3 + x^4/8|_0^2 = 12 - 32/3 + 210/3.\end{aligned}$$

(b) Find the volume of the region bounded by the planes $x = 3, x = 6 - y - z$, and $y = 0, z = 0$.

Solution: We have

$$\begin{aligned}V &= \int_0^3 \int_0^{3-y} \int_3^{6-y-z} 1\, dx\, dz\, dy \\ &= \int_0^3 \int_0^{3-y} 3 - y - z\, dz\, dy \\ &= \int_0^3 3z - yz - z^2/2|_0^{3-y}\, dy = \int_0^3 9 - 3y - (3y - y^2) - (9 - 6y + y^2)/2\, dy \\ &= \int_0^6 9/2 - 3y + y^2/2\, dy = 9y/2 - 3y^2/2 + y^3/6|_0^3 = -27/2 - 27/2 + 9/2 = 9/2.\end{aligned}$$