Math 212 Test 3 Solutions

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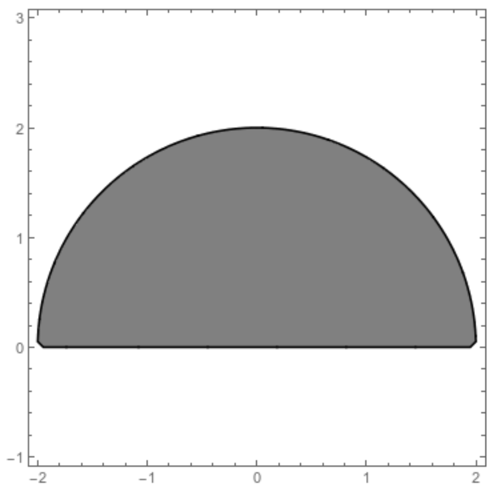
April 12, 2018

Problem 1. Let $f(x, y) = x^2 + y^2$ and let R be the semicircular region of radius 2 below.

(a) [15 points] Set up integrals to compute $\int_R f\, dA$ in both polar and cartesian coordinates.

(b) [15 points] Choose one of these integrals and evaluate it.



Solution:

(a)

$$\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} x^2 + y^2 \, dy \, dx$$

$$\int_0^{\pi} \int_0^{2} r^2 r \, dr \, d\theta$$

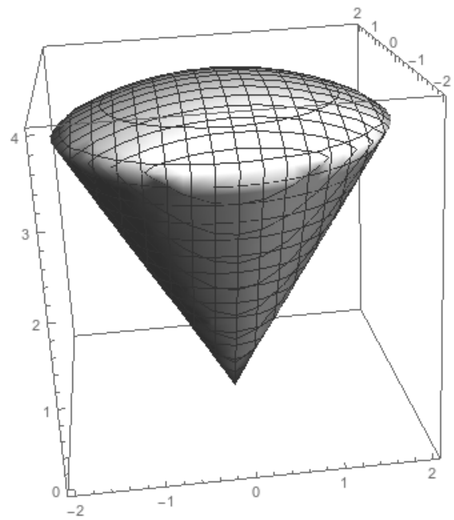
(b) We should use polar coordinates.

$$\begin{aligned} \int_0^{\pi} \int_0^{2} r^2 r \, dr \, d\theta &= \int_0^{\pi} \int_0^{2} r^3 \, dr \, d\theta \\ &= \int_0^{\pi} r^4/4|_0^2 \, dr \, d\theta = \int_0^{\pi} 4 \, d\theta \\ &= 4\theta_0^{\pi} = 4\pi. \end{aligned}$$

Problem 2. Let R be the spherical wedge bounded by a sphere of radius 4 centered at the origin, and the cone given by $z = \sqrt{3x^2 + 3y^2}$ (as shown below). Let $f(x, y, z) = z$.

(a) [15 points] Set up integrals to compute $\int_R f\, dA$ in cartesian, cylindrical, and spherical coordinates.

(b) [15 points] Choose one of these integrals and evaluate it.



Solution:

(a) We see that these intersect at the circle $x^2 + y^2 + 3x^2 + 3y^2 = 16$, or in other words $x^2 + y^2 = 4$, so the circle of radius 2 at the level $z = \sqrt{12} = 2\sqrt{3}$.

If we draw a triangle from the side, we see that we have a triangle with opposite side of length 2 and hypotenuse of length 4, so $\sin\phi = 1/2$. Thus $\phi = \pi/6$.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} z\,dz\,dy\,dx$$

$$\int_0^{2\pi}\int_0^2\int_{r\sqrt{3}}^{\sqrt{16-r^2}} zr\,dz\,dr\,d\theta$$

$$\int_0^4\int_0^{2\pi}\int_0^{\pi/6} \rho\cos\phi\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$

(b) I really hope everyone picks the spherical integral. We compute

$$\begin{aligned}
I &= \int_0^4\int_0^{2\pi}\int_0^{\pi/6} \rho^3\cos\phi\sin\phi\,d\phi\,d\theta\,d\rho \\
&= \frac{1}{2}\int_0^4\int_0^{2\pi} \rho^3\sin^2\phi|_0^{\pi/6}\,d\theta\,d\rho \\
&= \frac{1}{8}\int_0^4\int_0^{2\pi} \rho^3\,d\theta\,d\rho \\
&= \frac{\pi}{4}\int_0^4 \rho^3\,d\rho = \frac{\pi}{4}\frac{\rho^4}{4}|_0^4 = 16\pi.
\end{aligned}$$

Problem 3. (a) [10 points] Find a parametric equation for a particle moving in a straight line, starting at $(0, 0)$ and moving towards $(3, 2, 1)$.

(b) [10 points] Suppose another particle follows the path $\vec{r}_2(t) = (t^2, 9 - t, t)$. Does this particle's path intersect the path of the particle from part (a)?

(c) [10 points] Find a parametrization for the cone, opening in the direction of the x axis, with total inner angle $\pi/2$.

Solution:

(a) There are many correct answers, but one is $\vec{r}_1(t) = (3t, 2t, t)$.

(b) The paths intersect if and only if

$$3t_1 = t_2^2 \qquad 2t_1 = 9 - t_2 \qquad t_1 = t_2$$

The last equation tells us the times must be the same; then the first equation gives us that $t_2 = 3$ and the second equation also gives us that $t_2 = 3$. Thus they cross paths at $t_1 = t_2 = 3$.

(c) $\vec{r}(s,t) = (\sqrt{s^2+t^2}, s, t)$.
Alternatively, we can use cylindrical coordinates, and we have $\vec{r}(x,\theta) = (x, x\cos\theta, x\sin\theta)$.
Or we can use spherical coordinates, and get $\vec{r}(\rho,\theta) = (\rho\sqrt{2}/2, \rho\cos\theta\sqrt{2}/2, \rho\sin\theta\sqrt{2}/2)$.

Problem 4. (a) [15 points] Compute the integral of the function $f(x) = x + 3y$ over the region bounded by $x + 3y = 0, x + 3y = 3, x - 3y = 0, x - 3y = 2$. (Hint: reparametrize to get a rectangle).

(b) [5 points] Sketch the vector field $\vec{F}(x,y) = x\vec{i} + 1\vec{j}$.

(c) [10 points] Is $\vec{r}(t) = (e^t, t+1)$ a flow line for $\vec{F}$ from part (b)?

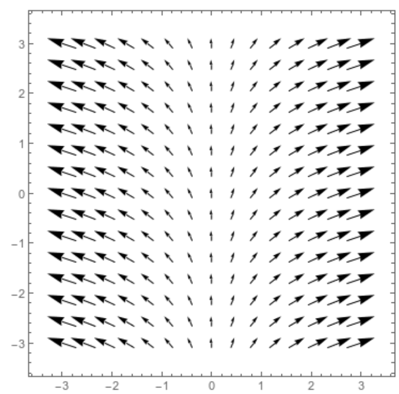
Solution:

(a) We use $s = x + 3y, t = x - 3y$. Then we have $x = \frac{s+t}{2}$ and $y = \frac{s-t}{6}$. Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/6 & -1/6 \end{vmatrix} = -1/12 - 1/12 = -1/6.$$

Thus our integral is

$$\int_0^3 \int_0^2 s|-1/6|\,dt\,ds = \int_0^3 \frac{s}{3}\,ds = \frac{s^2}{6}|_0^3 = \frac{3}{2}.$$

(b) 

(c) We have $\vec{F}(\vec{r}(t)) = e^t\vec{i} + \vec{j}$. We have $\vec{r}\,'(t) = (e^t, 1)$. Since these are the same, $\vec{r}$ is a flow line for $\vec{F}$.

Problem 5. (a) [15 points] Let $f(x,y) = xy$, and let C be the straight line segment from $(2,2)$ to $(3,5)$. Compute $\int_C f\,d\vec{r}$.

(b) [15 points] Let $\vec{F}(x,y) = xy\vec{i} + x\vec{j}$, and let C be parametrized by $\vec{r}(t) = (t^2, 3t)$ for $0 \le t \le 2$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

Solution:

(a) We can parametrize C with $\vec{r}(t) = (2+t, 2+3t)$ for $0 \le t \le 1$. Then

$$\begin{aligned} \int_C f\,d\vec{r} &= \int_0^1 (2+t)(2+3t)\sqrt{10}\,dt \\ &= \sqrt{10}\int_0^1 4 + 8t + 3t^2\,dt \\ &= \sqrt{10}(4t + 4t^2 + t^3)|_0^1 = \sqrt{10}(4+4+1) = 9\sqrt{10}. \end{aligned}$$

(b)

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^2 (3t^3, t^2)\cdot(2t, 3)\, dt \\ &= \int_0^2 6t^4 + 3t^2\, dt = \frac{6}{5}t^5 + t^3|_0^2 = \frac{192}{5} + 8 = \frac{232}{5} = 46.4.\end{aligned}$$