MATHEMATICS COMPREHENSIVE EXAMINATION, SPRING 2006

NAME: ______________________ Start Time: __________

Each question is worth 4 points. You have three hours. Show your work!

Single Variable Calculus

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1. Suppose you know that $f(t) = \int_1^t g(s)\,ds$ and that $g(1) = 2$.

   a. Find $f(1)$ and $f'(1)$.

   b. Find the equation for the line tangent to the graph of f at the point $(1, f(1))$.

$$f(1) = \int_1^1 g(s)ds = 0$$

$$\boxed{f'(1) = g(1) = 2}$$

$$y = f(1) + f'(1)(x-1)$$
$$= 0 + 2(x-1)$$
$$\boxed{y = 2x-2}$$

2. Consider the initial value problem $y' = -y^2$ and $y(0) = 2$.

   a. Use Euler's Method with stepsize $\Delta x = 1/2$ to estimate $y(1)$.

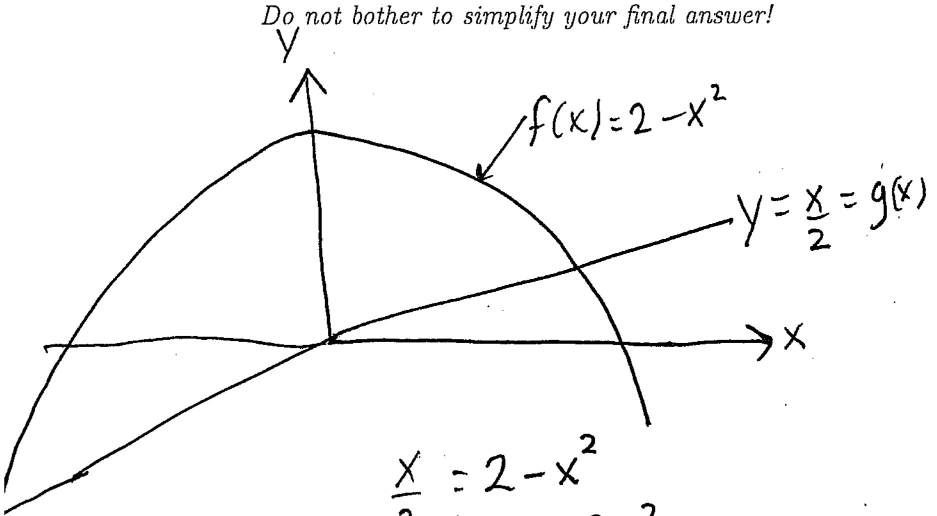
   b. Briefly explain how to obtain a better estimate of $y(1)$.

$$y(\tfrac{1}{2}) \approx y(0) + y'(0)\cdot\tfrac{1}{2}$$
$$= 2 + [-2^2]\cdot\tfrac{1}{2} = 2 - 2 = 0$$

$$y(1) \approx y(1/2) + y'(1/2)\cdot\tfrac{1}{2}$$
$$\approx 0 + [-0^2]\cdot\tfrac{1}{2} = 0$$

b. Use smaller Δx values to increase the accuracy of the approximation using Euler's Method

9. Find the *area* enclosed *between* the graph of the parabola $f(x) = 2 - x^2$ and the line $g(x) = \frac{1}{2}x$. *Do not bother to simplify your final answer!*



$$\text{Area} = \int_{\frac{-1-\sqrt{17}}{4}}^{\frac{-1+\sqrt{17}}{4}} 2 - x^2 - \frac{x}{2}\, dx$$

$$\frac{x}{2} = 2 - x^2$$
$$x = 4 - 2x^2$$
$$2x^2 + x - 4 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot -4}}{2(2)} = \frac{-1 \pm \sqrt{17}}{4}$$
$$(2x \quad)(x \quad)$$

10. Evaluate $\lim_{b \to \infty} \int_0^b e^{-x}\, dx$ $= \lim_{b \to \infty} -e^{-x}\Big|_0^b$

$$= \lim_{b \to \infty} -e^{-b} - (-e^0) = \lim_{b \to \infty} 1 - e^{-b}$$

$$= 1$$

5. Consider the function $h(x) = |x|$. For each of the following, either evaluate the expression or briefly explain why it doesn't exist:

a. $h'(0) =$ Does Not Exist because $h(x) = |x|$ has a "Kink" at $x = 0$.

b. $\int_{-1}^{1} h(x)\,dx =$

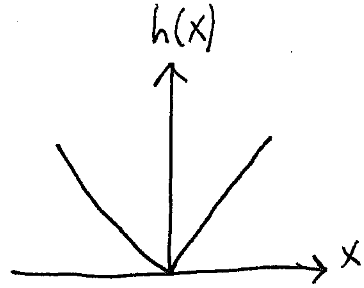
$$\int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1}$$

$$= -\frac{0^2}{2} - \left(-\frac{(-1)^2}{2}\right) + \frac{1^2}{2} - \frac{0^2}{2}$$

$$= +\frac{1}{2} + \frac{1}{2}$$

$$= 1$$



6. $\int x \ln x dx =$ $\int u\,dv = uv - \int v\,du = \frac{x^2}{2} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$

$u = \ln x \quad du = 1/x\,dx$

$dv = x \quad v = \frac{x^2}{2}$

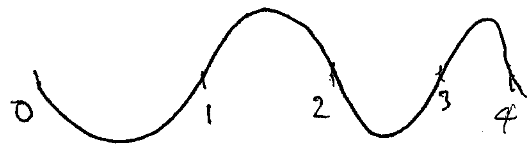
$$= \frac{1}{2}x^2 \ln x - \int \frac{x}{2}\,dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

7. A hiker starts hiking at 9 am. There are some hills; she walks faster downhill than uphill. Her *speed* is $s(t) = 1.5 - \sin(\pi t)$ miles per hour. Here t is *hours elapsed since 9 am.*

   a. How *far* did she hike by 1 pm?

   b. How many hills did she climb during that time? Explain.



$$s(t) = 1.5 - \sin(\pi t) = \frac{dD}{dt}$$

$$D(4) = \int_0^4 \frac{dD}{dt}\,dt = \int_0^4 s(t)\,dt = \int_0^4 1.5 - \sin(\pi t)\,dt$$

$$= 1.5t + \frac{\cos(\pi t)}{\pi}\Big|_0^4 = \left(1.5 \cdot 4 + \frac{\cos(4\pi)}{\pi}\right) - \left(1.5 \cdot 0 + \frac{\cos(0)}{\pi}\right)$$

$$= 6 + \frac{1}{\pi} - \frac{1}{\pi} = \boxed{6}$$

(b) 2 hills, since

8. Consider the infinite series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$.

   a. This series converges when $x = 1$. Why?

   b. This series is the Taylor series of a certain function $f(x)$ expanded about the origin. What is the value of $f''(0)$? Do *not* differentiate the series to find out!

a. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots =$ alternating Harmonic series, which converges

(b) Recall Taylor Series

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$$

$$\frac{f''(0)}{2!} = -\frac{1}{2} \Rightarrow \boxed{f''(0) = -1}$$

3. The following table of values for the *first* derivative $f'(x)$ is given:

x	0	1	2	3	4
$f'(x)$	4	3	0	-3	-4

a. Estimate the *second* derivative $f''(0)$. Show your work.

b. True or False (explain): The graph of the function f is concave up at $x = 0$.

a. $f''(0) \approx \dfrac{f'(1) - f'(0)}{1 - 0} \approx \dfrac{3-4}{1} \approx -1$

$\approx \dfrac{f'(4) - f'(0)}{4 - 0} \approx \dfrac{-4-4}{4} \approx -\dfrac{8}{4} = -2$

b. The derivative is decreasing, so the second derivative is negative and the curve would be concave down

4. Suppose $f(g(x)) = x$, $g(0) = 1$, and $f'(1) = 2$. Find $g'(0)$.

$f'(g(x)) \cdot g'(x) = 1$

$g'(0) = \dfrac{1}{f'(g(0))} = \dfrac{1}{f'(1)} = \dfrac{1}{2}$