PRACTICE Comprehensive Exam **Student Number AXXXXXXX:**

Department of Mathematics Tuesday, March 15, 2016

Closed book. Closed notes. NO CALCULATORS. Time allowed: 3 hours for 5 sections (proportionally less if taking fewer than 5 sections). In other words, 36 minutes for each section taken. Please write very legibly and cross out all scratch work.

Calculus 1

1. ______ 2. ______ 3. ______ 4. ______ 5. ______ Total: ______

Calculus 2

6. ______ 7. ______ 8. ______ 9. ______ 10. ______ Total: ______

Multivariable Calculus

11. ______ 12. ______ 13. ______ 14. ______ 15. ______ Total: ______

Linear Algebra

16. ______ 17. ______ 18. ______ 19. ______ 20. ______ Total: ______

Discrete Mathematics

21. ______ 22. ______ 23. ______ 24. ______ 25. ______ Total: ______

Multivariable Calculus

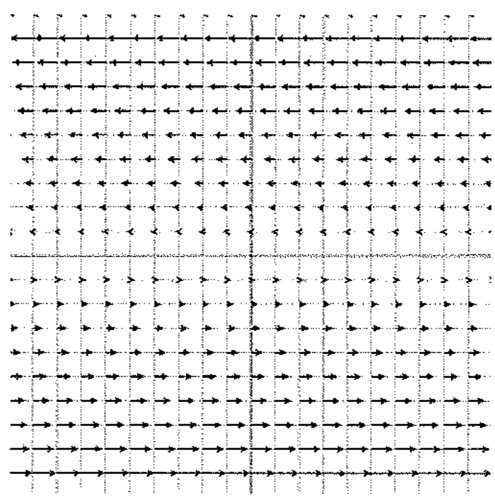
11. Evaluate the line integral along the curve C where $C: y^2 = x^3$ from $(1, -1)$ to $(1, 1)$.

$$\int_C (y - x)\, dx + x^2 y\, dy$$

$y = t^3$, $-1 \le t \le 1$
$x = t^2$, $-1 \le t \le 1$
$\frac{dy}{dt} = 3t^2$, $\frac{dx}{dt} = 2t$

$$\int_{-1}^{1} (t^3 - t^2)2t\,dt + t^7 \cdot 3t^2 dt$$
$$= \int_{-1}^{1} 3t^9 + 2t^4 - 2t^3\,dt$$
$$= \frac{3}{10}t^{10} + \frac{2}{5}t^5 - \frac{1}{2}t^4 \Big|_{-1}^{1}$$
$$= \left(\frac{3}{10} + \frac{2}{5} - \frac{1}{2}\right) - \left(\frac{3}{10} - \frac{2}{5} - \frac{1}{2}\right) = 4/5.$$

12. Consider the vector field $\vec{F}(x, y)$ below, estimate the signs of the following line integrals. EXPLAIN YOUR ANSWERS.



(a) $\int_{C_1} \vec{F} \cdot d\vec{x}$ where C_1 is the path along a circle of radius 1 centered at the origin, traversed in the counter-clockwise direction.

(b) $\int_{C_2} \vec{F} \cdot d\vec{x}$ where C_2 is the straight line path from $(1, 1)$ to $(-1, -1)$

(c) $\int_{C_3} \vec{F} \cdot d\vec{x}$ where C_2 is the straight line path from $(1, -1)$ to $(-1, -1)$

(d) $\int_{C_4} \vec{F} \cdot d\vec{x}$ where C_2 is the straight line path from $(1, -1)$ to $(1, 1)$

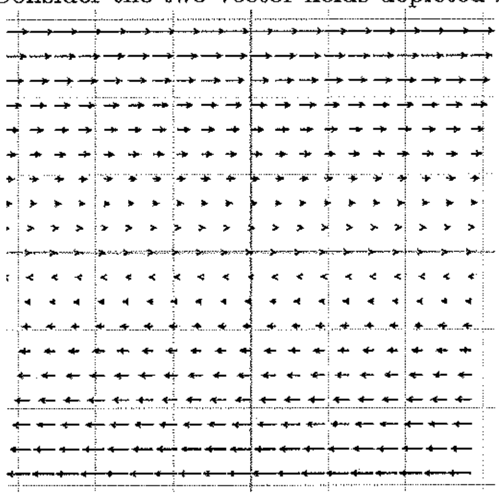
a) positive, always aligned w/ arrows

b) Zero, symmetric

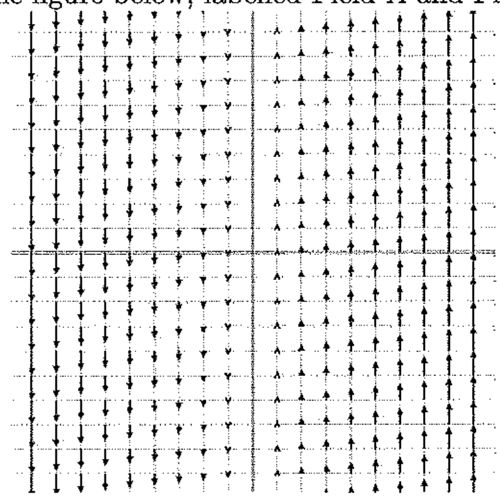
c) ~~Zero, symmetric~~ Negative, going backwards against field

d) Zero, perpendicular to v. field

13. Consider the two vector fields depicted in the figure below, labelled Field $\vec{A}$ and Field $\vec{B}$.



FIELD $\vec{A}$



FIELD $\vec{B}$

One of these two vector fields is a conservative field, i.e. can be written as the gradient of some function $\vec{\nabla} f$. Identify the conservative field, explain your selection and find the potential function $f(x, y)$ which produces the vector field you have selected.

Neither

$\vec{A} = (y, 0)$

If $\nabla f = \vec{A}$

then $f_x = y \Rightarrow f = xy + g(x)$

$f_y = 0 \Rightarrow f = g(x) \Rightarrow \Leftarrow$

$\vec{B} = (0, x)$

if $\nabla f = \vec{B}$

$f_x = 0 \Rightarrow f(x,y) = g(y)$

$g'(y) = x \Rightarrow g(y) = xy + C \Rightarrow \Leftarrow$

14. Find the critical points of $f(x, y) = x^2y + 2y^2 - 2xy + 6$ and use the second derivative test to classify them. (You do NOT need to identify global extrema of this function.)

$\nabla f(x,y) = (2xy - 2y,\ x^2 + 4y - 2x) = (0, 0)$

$2y(x-1)$ $\qquad$ $y = 0$: $x = 0$ or $x = \cancel{\pm\sqrt{3}}\ 2$

$y = 0$ or $x = 1$ $\qquad$ $x = 1$: $y = 1/4$

CP are $(0,0)$, ~~$(\sqrt{3}, 0), (-\sqrt{3}, 0)$~~, $(1, 1/4)$, $(2, 0)$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (2y)(4) - (2x-2)^2 = 8y - (4x^2 - 8x + 4)$

$D(0,0) = -4 < 0$: saddle point

$D(2,0) = -4 < 0$: saddle point

~~$D(1, 1/4) = 2 - (\frac{1}{4} - 2 + 4) = -1/4 < 0$: saddle point~~

$D(1, 1/4) = 2 - (4 - 8 + 4) = 2 > 0$, $f_{xx}(1, 1/4) = 1/2 > 0$ local min

15. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

$g(x,y,z) = x + y + z = 27$

$f(x,y,z) = x^2 + y^2 + z^2$

$\nabla f(x,y,z) = (2x, 2y, 2z)$

$\nabla g(x,y,z) = (1,1,1)$

$$\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned} \Rightarrow x = y = z = \lambda/2$$

$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 27$

$3\lambda = 54$

$\lambda = 18$

$x = y = z = 9$

16. Find the global maximum and global minimum of the function $f(x, y) = 4x + 2y$ on the region $x^2 + 4y^2 \leq 9$, if they exist.

f cts on closed bdd region, global extrema exist by EVT

$\nabla f(x,y) = (4,2) \neq 0$ so extrema lie on bdry

$g(x) = x^2 + 4y \leq 9$

$\nabla g(x) = (2x, 8y)$

$$\cancel{\begin{aligned} y &= 2x\lambda \\ 2 &= 8y\lambda \end{aligned} \Rightarrow \begin{aligned} x &= 2/\lambda \\ y &= \tfrac{1}{4\lambda} \end{aligned}}$$

$$\begin{aligned} 4\lambda &= 2x \\ 2\lambda &= 8y \end{aligned} \Rightarrow \begin{aligned} x &= 2\lambda \\ y &= \lambda/4 \end{aligned}$$

$(2\lambda)^2 + 4(\lambda/4)^2 = 9$

$4\lambda^2 + \lambda^2/4$

$17\lambda^2 = 36$

$\lambda = \pm \frac{6}{\sqrt{17}}$

2 cases:

A) $x = \frac{12}{\sqrt{17}}$, $y = \frac{3}{2\sqrt{17}}$ $\quad f(x,y) > 0$

global max

B) $x = \frac{-12}{\sqrt{17}}$, $y = \frac{-3}{2\sqrt{17}}$ $\quad f(x,y) < 0$

global min

17. Find a point on the surface $z = 3x^2 - y^2$ at which the tangent plane is parallel to the plane $6x + 4y - z = 5$.

$$3x_0^2 - y_0^2$$
$$f(x_0, y_0)$$
$$T_2(x,y) = f(x_0,y_0) + 6x_0(x - x_0) - 2y_0(y - y_0)$$

normal vector $(6x_0, -2y_0)$

$z = 6x + 4y + 5$ has normal vector $6, 4$

$$6x_0 = 6 \qquad x_0 = 1$$
$$\Rightarrow$$
$$-2y_0 = 4 \qquad y_0 = -2$$

18. Show that the surfaces

$$z = \sqrt{x^2 + y^2}$$

and

$$z = \frac{1}{10}(x^2 + y^2) + \frac{5}{2}$$

intersect at $(3, 4, 5)$ and have a common tangent plane at that point.

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5 \checkmark$$
$$\frac{1}{10}(3^2 + 4^2) + 5/2 = \frac{1}{10}(25) + \frac{25}{10} = \frac{50}{10} = 5 \checkmark$$

$(3,4,5)$ is on both surfaces, so they intersect there.

$$f(x,y) = \sqrt{x^2+y^2}, \quad f_x = \frac{x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2}}$$
$$T_f(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

$$g(x,y) = \frac{1}{10}(x^2+y^2) + 5/2$$
$$g_x = \frac{x}{5}, \quad g_y = \frac{y}{5}$$
$$T_g(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

are the same plane

19. Let G be the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the coordinate planes.

$$\mathcal{I} = \iiint_G xyz\, dV$$

(a) Write down a definite integral to evaluate $\mathcal{I}$ using two different coordinate systems. rectangular coordinates;

(b) Evaluate one of your integrals to obtain the exact value of $\mathcal{I}$.

a) $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz\, dz\, dy\, dx$

spherical: $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \sin\varphi \left(\rho \sin\varphi \cos\theta\ \rho \sin\varphi \sin\theta\ \rho \cos\varphi\right) d\rho\, d\theta\, d\varphi$

b) $= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^5 (\sin^3\varphi \cos\varphi)(\sin\theta\cos\theta)\, d\rho\, d\varphi\, d\theta$

$= \int_0^{\pi/2} \int_0^{\pi/2} (\sin^3\varphi \cos\varphi)(\sin\theta \cos\theta) \frac{\rho^6}{6}\Big|_0^2 d\varphi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \frac{32}{3} (\sin^3\varphi \cos\varphi)(\sin\theta\cos\theta)\, d\varphi\, d\theta$

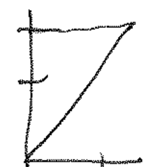
$= \frac{32}{3} \int_0^{\pi/2} (\sin\theta \cos\theta) \frac{\sin^4\varphi}{4}\Big|_0^{\pi/2} d\theta = \frac{32}{3} \int_0^{\pi/2} \sin\theta\cos\theta \frac{1}{4}\, d\theta$

$= \frac{8}{3}\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right) = \frac{4}{3}.$

20. Evaluate the integral by first reversing the order of integration.

$$\int_0^2 \int_{y/2}^1 \cos(x^2)\, dx\, dy$$

Region



$\int_0^1 \int_0^{2x} \cos(x^2)\, dy\, dx$

$= \int_0^1 y\cos(x^2)\Big|_0^{2x} dx$

$= \int_0^1 2x\cos(x^2)\, dx$

$= \sin(x^2)\Big|_0^1 = \sin(1) - \sin(0)$

$= \sin(1)$