Statement of Research Interests

Jay Daigle

1 Introduction

My research focuses on the subject of number theory, and in particular using algebraic tools and various group representations to study number-theoretic questions. I am currently working on two research projects: one using (ϕ, Γ)-modules to study the equivariant Tamagawa number conjecture, and one using the recently-developed tools of supercharacter theory to study number-theoretic identities.

2 The equivariant Tamagawa number conjecture

2.1 An overview of the Tamagawa number conjecture

The Tamagawa number conjecture of [BK90] is a deep conjecture about special values of L-functions, which relates to both the Birch and Swinnerton-Dyer conjecture and the Riemann zeta function (and in fact the Birch and Swinnerton-Dyer conjecture is a special case of the Tamagawa number conjecture). The conjecture also has close ties to the main conjecture of Iwasawa theory.

Given a "motive" M (which can be thought of as a smooth projective variety X over $\mathbb{Q}$ with some cohomological data attached), we can write an L-series

$$L(M, s) = \prod_p P_p(p^{-s})^{-1} \tag{1}$$

where $P_p(T) = \det(1 - Fr_p^{-1} \cdot T)$ (where Fr_p is the Frobenius operator on a certain space) is a polynomial, conjectured to lie in $\mathbb{Q}[T]$. This L-function is a complex series which is conjectured (and known in many cases) to converge for $\Re(s) \gg 0$. The Tamagawa number conjecture holds that $L(M, s)$ has a meromorphic continuation to the complex plane, and predicts the behavior at zero. That is, we can write a Laurent series centered at zero

$$L(M, s) = L^*(M)s^{r(M)} + \dots$$

and the Tamagawa number conjecture predicts $L^*(M) \in \mathbb{R}$ and $r(M) \in \mathbb{Z}$.

If L is a number field and $M = h^0(L)(0)$, then $L(M, s)$ is the Dedekind zeta function

$$L(M, s) = \zeta_L(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} (1 - N_{K/\mathbb{Q}}(P))^{-s}$$

Then the Tamagawa number conjecture is equivalent to the analytic class number formula: if $L \otimes_{\mathbb{Q}} \mathbb{R} \cong R^{r_1} \times \mathbb{C}^{r_2}$ then $r(M) = r_1 + r_2 - 1$ and $L^*(M) = -hR/w$ where R is the regulator of L, w the number of roots of unity in L, and h is the class number of $\mathcal{O}_L$.

If X is an abelian variety over a number field and $M = h^1(X)(1)$ then $L(M, s-1)$ is the classical Hasse-Weil L-function of the dual abelian variety and thus also of X. In particular, if X is an elliptic curve over $\mathbb{Q}$ then the Tamagawa number conjecture implies the Birch and Swinnerton-Dyer conjecture, which holds that $r(M)$ is the rank of the elliptic curve and $L^*(M)$ is given by various arithmetic data associated to X.

The conjecture of [BK90] was later refined by [FK06] and others to an equivariant statement, using algebraic K-theory, which can incorporate information from a $\mathbb{Q}$-algebra acting on M. We can investigate this equivariant conjecture "locally," on each of the P_p factors of the L-function.

We are far from being able to prove the equivariant Tamagawa number conjecture in full generality, but progress has been made on easier cases. The analytic class number formula has been known at least since Dirichlet, and the Birch and Swinnerton-Dyer conjecture is an area of active research with many partial results; in their original Tamagawa number conjecture paper, [BK90] show that the Tamagawa number conjecture, and thus the l-primary part of the Birch and Swinnerton-Dyer conjecture, holds for elliptic curves with CM. They also spend some time addressing a motive known as the Tate motive.

2.2 The equivariant Tamagawa number conjecture for the Tate motive

The Tate motive $\mathbb{Q}(r)$, is the formal inverse to the Lefschetz motive attached $\mathbb{P}^1$. It is associated to the classical Riemann zeta function, since its L-function over $\mathbb{Q}$ is $L(s, \mathbb{Q}(r)) = \zeta(s+r)$. It is difficult to define in generality, but the Tate motive over the field $\mathbb{Q}_p$ is simply $\mathbb{Q}_p$ acted on by the rth power of the cyclotomic character—that is, if $\chi^{\text{cyclo}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$ is the cyclotomic character, then $\mathbb{Q}_p(r)$ has the action $\gamma \cdot x = \chi^{\text{cyclo}}(\gamma)^r \cdot x$.

The Tamagawa conjecture for the Tate motive over a finite extension $K/\mathbb{Q}_p$ can be stated:

Conjecture 2.1 (local Tamagawa number conjecture for Tate motives)**.** *There are elements $\epsilon(K/\mathbb{Q}_p, 1-r), [C_\beta] \in K_1(\overline{\mathbb{Q}_p}[\mathrm{Gal}(K/\mathbb{Q}_p)])$ such that if $\beta \in H^1(K, \mathbb{Z}_p(1-r))$ spans a free $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q}_p)]$-submodule, then the element*

$$[(r-1)!] \cdot \epsilon(K/\mathbb{Q}_p, 1-r) \cdot [\mathrm{per}(\exp^*(\beta) \otimes 1)] \cdot [C_\beta] \cdot \left[\frac{1-p^{r-1}\phi}{1-p^{-r}\phi^{-1}}\right] \tag{2}$$

of $K_1(\mathbb{Q}_p^{ur}[\mathrm{Gal}(K/\mathbb{Q}_p)])$ lies in the image of $K_1(\mathbb{Z}_p^{ur}[\mathrm{Gal}(K/\mathbb{Q}_p)])$.

The case where $r \geq 2$ and K is an unramified extension of $\mathbb{Q}_p$ was proven in [BK90] modulo a power of 2 (and modulo an additional conjecture if r is odd). The proof has three ingredients. First, there is a reciprocity law involving the Bloch-Kato exponential, allowing explicit computations of the map $\exp : K \tilde{\to} H^1(K, \hat{\mathbb{Z}}(r)) \otimes \mathbb{Q}$. Second, there is the Coleman exact sequence

$$0 \to \mathbb{Z}_p(1) \to U' \to^{\phi} \mathcal{O}_K[\![\pi]\!]^{\psi=0} \to \mathbb{Z}_p(1) \to 0, \tag{3}$$

where U' is the pro-p part of $U = \varprojlim_n (\mathcal{O}_K[\zeta_{p^n}]^\times)$. Third, there is a basis for the space $\mathcal{O}_K[\![\pi]\!]^{\psi=0}$ over $\mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$: specifically, $\mathcal{O}[\![\pi]\!]^\psi$ is a rank-1 $\mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$-module (see [PR90]).

There are a few alternate proofs of the reciprocity law of [BK90] , but all of them depend on the exact sequence (3), and neither this exact sequence nor the reciprocity law have easy or obvious generalizations to the case where K is ramified over $\mathbb{Q}_p$.

2.3 Using (ϕ, Γ)-modules

In my thesis I constructed a new and more easily generalizable proof which uses neither the reciprocity law nor the Coleman exact sequence. Instead, I use a reciprocity law from [CC99] to translate conjecture 2.1 into a statement about the image of a certain (ϕ, Γ)-module under the dual map to the Perrin-Riou exponential:

Theorem 2.2 (Cherbonnier-Colmez IV.2.1)**.** *Let p be a prime and let $K/\mathbb{Q}_p$ be finite. Let $\mathrm{Exp}^*_{\mathbb{Q}_p} : H^1_{Iw}(K, \mathbb{Q}_p(1)) \tilde{\to} A_K^{\psi=1}(1)$ be the inverse of the Perrin-Riou exponential. Let T_m be defined for any $K(\zeta_{p^n})$ as $p^{m-n}\,\mathrm{Tr}_{K(\zeta_{p^n})/K(\zeta_{p^m})}$ and extend it linearly to $K[\![t]\!]$. Then*

1. *If n is large enough and $\mu \in H^1_{Iw}(K, \mathbb{Q}_p(1))$, then $T_m(\phi^{-n}(\mathrm{Exp}^*_{\mathbb{Q}_p}(\mu)))$ is an element of $K(\zeta_{p^m})((t))(1)$ independent of n.*
2. *If n is large enough and $\mu \in H^1_{Iw}(K, \mathbb{Q}_p(1))$, then*

$$(T_m \circ \phi^{-n})\left(\mathrm{Exp}^*_{\mathbb{Q}_p}(\mu)\right) = \sum_{r \geq 0} \exp^*_{\mathbb{Q}_p(1-r)} pr_{m,r}(\mu) \otimes t^{r-1} \tag{4}$$

where $pr_{m,r}$ is the projection of the Iwasawa cohomology onto its mth component, with a twist by r.

This reciprocity law does not require that $K/\mathbb{Q}_p$ be unramified, but holds for arbitrary finite extensions of $\mathbb{Q}_p$. Since this relates Exp^* on $H^1_{Iw}(K, \mathbb{Q}_p(1))$ to $\exp^*$ on $H^1(K, \mathbb{Q}_p(1-r))$, I used it to prove:

Proposition 2.3. *1. The ring $A_K^{\psi=1}(1)$ mentioned in Theorem 2.2 is a rank one $\Lambda_K = \mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q}_p)] \otimes_{\mathbb{Z}_p} \Lambda$-module, where Λ is the Iwasawa algebra.*

2. If α is spans a free Λ_K-submodule of $A_K^{\psi=1}(1)$, and for some $m \geq 0$ β spans a free $\mathbb{Z}_p[\mathrm{Gal}(K(\zeta_{p^m})/\mathbb{Q}_p)]$-submodule of $H^1(K(\zeta_{p^m}), \mathbb{Z}_p(1-r))$ as in theorem 2.2, then

$$\exp^*(\beta) = \left(\frac{d^{r-1}}{dt^{r-1}} T_m \phi^{-n} \frac{\alpha}{(r-1)!}\right)\Bigg|_{t=0}. \tag{5}$$

Though this result is technical and uninteresting in itself, it turns out that if α is known, the quantity on the right-hand side of (5) is easy to calculate if you know α explicitly. A generalization by [PR90] of the Coleman exact sequence allows us to calculate α for $K/\mathbb{Q}_p$ unramified, and thus proves:

Theorem 2.4. *If $r \geq 2$, $p > 2$, and $K/\mathbb{Q}_p$ unramified, then the the local Tamagawa number conjecture holds for the Tate motive $\mathbb{Q}_p(r)$ over K and over $K(\zeta_p)$.*

This gives a new proof of the result of [BK90], and easily extends to the tamely ramified case of $K(\zeta_p)$. I have not yet extended this method to $K(\zeta_{p^m})$ for any $m > 0$, but the difficulty lies almsot entirely in computing the appropriate $\epsilon(K(\zeta_{p^m})/\mathbb{Q}_p, 1-r)$; this method easily determines the actual element described in (2). I am also working with Matthias Flach to prove conjecture 2.1 for any tamely ramified extension $K/\mathbb{Q}_p$, but while we are confident in our approach the details are not quite complete.

2.4 Research Goals

My most immediate research goal is to finish our extension of Theorem 2.4 to all tamely ramified finite extensions $K/\mathbb{Q}_p$. Perrin-Riou's generalization of the Coleman sequence does not allow us to compute an exact α, but a careful combinatorial argument on $A_K^{\psi=1} \mod p$ seems to show that the conjecture holds.

In the medium term, I would like to extend our results to $p = 2$; this is not in principle difficult, but requires dealing with some unpleasant details which are not present in the case where p is odd. I also plan to work on computing the $\epsilon(K(\zeta_{p^m})/\mathbb{Q}_p, 1-r)$ factor for $m > 1$. This would allow us to prove conjecture 2.1 for cyclotomic extensions of unramified extensions of $\mathbb{Q}_p$, and possibly of tamely ramified extensions as well.

In the long term, I'd like to use these and similar techniques to study other motives and their L-functions as well. The reciprocity law of Theorem 2.2 applies to any Galois representation, not just $\mathbb{Q}_p(r)$, and thus can be used to study motives other than the Tate motive. The field of (ϕ, Γ)-modules is still developing and I hope to sharpen these tools, and apply them to reveal more about L-functions and the Tamagawa number conjecture.

3 Supercharacter theory

Supercharacter theory is a new approach to studying group characters, recently developed by André and Diaconis-Isaacs among others ([And02, And95, DI08]). More recently, it has been shown that a number of number-theoretic exponential sums arise naturally from supercharacter theories on the group $(\mathbb{Z}/n\mathbb{Z})^d$. In particular Ramanujan sums, Kloosterman sums, Gauss sums, and Heilbronn sums can all be described this way ([BBF$^+$14]).

Viewing these sums as supercharacters directly yields a number of non-trivial classical identities, and almost every algebraic identity involving Ramanjuan sums has been rederived from a supercharacter theory approach ([FGK13]). We can derive bounds on various exponential sums, including recovering the Heath-Brown bound on Heilbronn sums, and we can relate exponential sums to finding the number of solutions of a system of algebraic congruences, which is a long-standing problem in arithmetic geometry ([GHL13]). Further, plotting the images of these supercharacters produces pictures with interesting and unusual symmetries (and, perhaps more interestingly, pseudo-symmetries) ([DGL12, BBGG$^+$13].

I am working with a group at Pomona College to better understand the graphical properties of supercharacter theories. This is a field rich with possibilities and easily accesible for joint work with enthusiastic undergraduates, which also has deep ties to bounds on the magnitude of exponential sums.

I am also working to extend supercharacter theory to a broad realm of applicability. Supercharacters have only been developed for complex characters, but I have shown that much of the groundwork can also be implemented for characters into other algebraically closed fields. It is known from [BBF$^+$14] and [FGK13] that complex supercharacter theories on finite groups have a well-defined Fourier theory, and I am working to extend that theory to non-complex characters and to all self-Pontryagin-dual groups.

References

[And95] Carlos AM André, *Basic characters of the unitriangular group*, Journal of algebra **175** (1995), no. 1, 287–319.

[And02] Carlos André, *Basic characters of the unitriangular group (for arbitrary primes)*, Proceedings of the American Mathematical Society **130** (2002), no. 7, 1943–1954.

[BBF+14] J.L. Brumbaugh, Madeleine Bulkow, Patrick S. Fleming, Luis Alberto Garcia German, Stephan Ramon Garcia, Gizem Karaali, Matt Michal, Andrew P. Turner, and Hong Suh, *Supercharacters, exponential sums, and the uncertainty principle*, Journal of Number Theory **144** (2014), no. 0, 151 – 175.

[BBGG+13] JL Brumbaugh, Madeleine Bulkow, Luis Alberto Garcia German, Stephan Ramon Garcia, Matt Michal, and Andrew P Turner, *The graphic nature of the symmetric group*, Experimental Mathematics **22** (2013), no. 4, 421–442.

[BK90] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400. MR 1086888 (92g:11063)

[CC99] Frédéric Cherbonnier and Pierre Colmez, *Théorie d'Iwasawa des représentations p-adiques d'un corps local*, J. Amer. Math. Soc. **12** (1999), no. 1, 241–268. MR 1626273 (99g:11141)

[DGL12] William Duke, Stephan Ramon Garcia, and Bob Lutz, *The graphic nature of gaussian periods*, arXiv preprint arXiv:1212.6825 (2012).

[DI08] Persi Diaconis and I Isaacs, *Supercharacters and superclasses for algebra groups*, Transactions of the American Mathematical Society **360** (2008), no. 5, 2359–2392.

[FGK13] Christopher F. Fowler, Stephan Ramon Garcia, and Gizem Karaali, *Ramanujan sums as supercharacters*, The Ramanujan Journal (2013), 1–37 (English).

[FK06] Takako Fukaya and Kazuya Kato, *A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory*, Proceedings of the St. Petersburg Mathematical Society. Vol. XII, Amer. Math. Soc. Transl. Ser. 2, vol. 219, Amer. Math. Soc., Providence, RI, 2006, pp. 1–85. MR 2276851 (2007k:11200)

[GHL13] Stephan Ramon Garcia, Mark Huber, and Bob Lutz, *Algebraic properties of heilbronn's exponential sum: supercharacters, fermat congruences, and heath-brown's bound*, arXiv preprint arXiv:1312.1034 (2013).

[PR90] Bernadette Perrin-Riou, *Théorie d'Iwasawa p-adique locale et globale*, Invent. Math. **99** (1990), no. 2, 247–292. MR 1031902 (91b:11116)