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Non-Minimal Factorization in Numerical Monoids

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Introduction

A **monoid** is a set M with a binary associative operation $*$ and an identity element, 1. That is, for all $a, b \in M$, we have

1. $a * b \in M$
2. $a * (b * c) = (a * b) * c$
3. $1 * a = a * 1 = 1$.

Numerical Monoids

A **Numerical Monoid** is an additive submonoid of $(\mathbb{N}, +)$. We say a numerical monoid M is generated by $S = \{n_1, \dots, n_k\}$ (write $M = \langle n_1, \dots, n_k \rangle$) if M is the smallest numerical monoid containing every element of S. S is a minimal generating set if no proper subset of S also generates M. It turns out that every numerical monoid has a unique minimal generating set.

A numerical monoid is **primitive** if the GCD of the generators is 1. Every numerical monoid is isomorphic to a primitive numerical monoid, so we may ignore the rest. A primitive monoid contains all but a finite subset of the natural numbers; the largest natural number not in the monoid is the **Frobenius Number**.

The Fundamental Theorem of Arithmetic says that in the monoid $(\mathbb{N}, \cdot)$ every element has exactly one factorization into irreducible elements, but this result does not generalize, and in particular it does not hold in numerical monoids. In the monoid $\langle 5, 7 \rangle$,

$$50 = 10 \cdot 5 = 3 \cdot 5 + 5 \cdot 7$$

and thus the element 50 has two different factorizations, of different lengths. We define the **set of lengths** of x, $\mathcal{L}(x)$, to be the set of lengths of possible factorizations of x; thus $\mathcal{L}(50) = \{8, 10\}$.

Length sets and Delta sets

The **delta set** of x is the set of consecutive differences in $\mathcal{L}(x)$. That is, if $\mathcal{L}(x) = \{n_1, \dots, n_k\}$ with $n_1 \leq \cdots \leq n_k$, then

$$\Delta(x) = \{n_2 - n_1, n_3 - n_2, \dots, n_k - n_{k-1}\}.$$

In our example, we have $\Delta(50) = \{2\}$.

Finally, we define

$$\Delta(M) = \bigcup_{x \in M} \Delta(x).$$

Non-Minimal Generators

All these definitions deal with factorizations into irreducible elements, but we can factor with respect to any set that generates our monoid. Formally, let S generate a numerical monoid M. Then we define the set of lengths of x with respect to S, $\mathcal{L}^S(x)$, and the delta set with respect to S, Δ^S, by analogy with the usual definitions. Thus in the monoid $M = \langle 5, 7 \rangle$, if $S = \{5, 7, 12\}$ we have

$$\begin{aligned} 50 &= 2 \cdot 7 + 3 \cdot 12 = 1 \cdot 5 + 3 \cdot 7 + 2 \cdot 12 \\ &= 2 \cdot 5 + 4 \cdot 7 + 1 \cdot 12 = 3 \cdot 5 + 5 \cdot 7 = 10 \cdot 5 \end{aligned}$$

so $\mathcal{L}^S(50) = \{5, 6, 7, 8, 10\}$ and $\Delta^S(50) = \{1, 2\}$.

Some well-known results in the minimal case follow immediately with the new definitions:

- **Proposition:** Let $S = \{n_1, \dots, n_k\}$. Then $\min(\Delta^S(M)) = \gcd(\Delta^S(M)) = \gcd\{n_i - n_{i-1}\}$.

A Few Examples

Nice Delta Sets

Certain classes of numerical monoids have extremely nice delta sets.

- Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, n_1 + n_2\}$. Then $\Delta^S(M) = \{1, 2, \dots, n_2 - n_1\}$.
- Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, n_1 n_2\}$. Then $\Delta^S(M) = \Delta^S\left(\left(\frac{n_2 - 1}{\gcd(n_1 - 1, n_2 - n_1)} n_1 n_2\right)\right)$.

Less Nice Delta Sets

But not all obtainable delta sets are nice-looking. Some have large gaps, multiple gaps, and sometimes very strange apparent structures.

- $M = \langle 3, 8 \rangle, S = \{3, 8, 96\}, \Delta^S(M) = \{1, 2, 3, 4, 5, 6, 11\}$.
- $M = \langle 6, 11 \rangle, S = \{6, 11, 48\}, \Delta^S(M) = \{1, 2, 3, 5, 7\}$.
- $M = \langle 6, 11, 49 \rangle, S = \{6, 11, 49\}, \Delta^S(M) = \{1, 2, 3, 5, 8, 11\}$

Adding Many Elements

Is there a limit to how simple or ugly we can make a delta set by adding additional generators? In both cases, the answer is no.

Theorem: Let M be a numerical monoid, and $\{n_1, \dots, n_k\}$ be a generating set for M.

For all $N \geq \left\lceil \frac{n_k}{n_1} \right\rceil n_k$, if $S = \{m \in M \mid m \leq N\}$, then $\Delta^S(M) = \{1\}$.

Theorem: For any numerical monoid M and any $n \in \mathbb{N}$, there is a finite generating set S such that $|\Delta^S(M)| > n$.

Adding One Element

Next we asked what could happen if we took a monoid generated by two elements and added one extra generator. To analyze this setup we proved a corollary to a theorem by Nathan Kaplan:

Theorem: Let $M = \langle n_1, n_2, n_3 \rangle$ be a numerical monoid with $n_1 < n_2 < n_3$. Then $\max(\Delta(M)) = \max(\Delta(k_1 n_1) \cup \Delta(k_3 n_3))$, where $k_1 = \min\{k \mid k n_1 \in \langle n_2, n_3 \rangle\}$ and $k_3 = \min\{k \mid k n_3 \in \langle n_1, n_2 \rangle\}$.

Corollary: Let $M = \langle n_1, n_2 \rangle$ be a numerical monoid, and let $S = \{n_1, n_2 s\}$, with $s = i n_1 + j n_2$. Then

1. If $j \neq 0$, $\max(\Delta^S(H)) = \max\{n_2 - n_1, i + j - 1\}$.
2. If $j = 0$ and $n_2 < s$, $\max(\Delta^S(H)) = i - 1$.
3. If $j = 0$ and $s < n_2$, $\max(\Delta^S(H)) = \max\{i - 1, \lceil n_2 / i \rceil + \lfloor n_2 / i \rfloor - n_1\}$.

Leaving the Delta Set Unchanged

First we tried to see how small a change we could get adding one element—could we keep the delta set down to one element, or even leave it unchanged? The answer is yes to both.

Theorem: Let M and S be as above. Then $\Delta(M) = \Delta^S(M)$ if and only if $i + j - 1 = n_2 - n_1$.

Theorem: Let M and S be as above. Then $|\Delta^S(M)| = 1$ if and only if one of the following two conditions holds:

1. $i + j - 1 = n_2 - n_1$.
2. $j = 0$ and $m(i + j - 1) = n_2 - n_1$ such that $m \leq \lceil n_2 / i \rceil$.

Avoiding Holes

Sampling of various delta sets suggests that "most" delta sets are nice. We conjecture that if S is any set of natural numbers and $M = \langle S \rangle$, then $\Delta^S(M) = \{1, k\}$ only if $k = 1$ or $k = 2$. That is, no monoid has the delta set $\{1, 3\}$. But it's difficult to prove that a given set is never a delta set. However, we were able to prove a limited case, and along the way discover another class of monoids and generating sets with particularly nice delta sets.

Proposition: Let $M = \langle n_1, n_2 \rangle$ be a primitive monoid, and let $S = \{n_1, n_2, i n_1 + j n_2\}$. Suppose $i + j = 2$. Then $\Delta^S(M) = [1, k]$ for some k.

Proposition: Let $M = \langle n_1, n_2 \rangle$ be a primitive monoid and let $i, j \in \mathbb{N}_0$ such that $i + j - 1 = k(n_2 - n_1) = k\alpha$ for some $k > 0$. Then if $S = \{n_1, n_2, i n_1 + j n_2\}$, $\Delta^S(M) = \{\alpha, 2\alpha, \dots, k\alpha\}$.

Theorem: Let n_1, n_2 be positive relatively prime integers, and let $M = \langle n_1, n_2 \rangle$. Let $i, j \in \mathbb{N}_0$, and let $S = \{n_1, n_2, i n_1 + j n_2\}$. Then if $\Delta^S(M) = \{1, k\}$, $k = 2$.

Periodicity

We note that we can find the delta set of the whole monoid by examining delta sets of all elements up to a certain point, and then applying the following:

Theorem: Let M be a numerical monoid generated by $S = \{n_1, n_2, \dots, n_k\}$. Then if $x \geq 2 k n_2 n_k^2$, we have $\Delta^S(x) = \Delta^S(x + n_1 n_k)$.

Open Questions

- The periodicity bound $2 k n_2 n_k^2$ is far from optimal in most cases. It might be possible to prove a much better bound, if only in more limited cases, like $k = 3$.
- It seems that every element in $\Delta(M)$ shows up in $\Delta(x)$ for infinitely many x. This is proved if the following holds:
  Conjecture: For all x, $\Delta(x) \subset \Delta(x + n_1 n_k)$.
- Can we get similar results for other classes of monoids (half-factorial, block monoids, etc.)?

References

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For Further Information

Details of these proofs and related work, as well as this poster, are available at `http://www.dci.pomona.edu/~jadagul`. Correspondence can be directed to `gerald.daigle@pomona.edu` or `hoyerrol@grinnell.edu`.

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