Math 1232 Midterm Solutions

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March 1 2022

1. You will have 75 minutes for this test.
2. You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
3. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.
4. You should compute all trigonometric, inverse trigonometric, exponential, and logarithmic functions as far as it is possible to do by hand. Outside of approximation questions I prefer not to see decimal answers.
5. This test has eight questions, over five pages. You do not have to answer all eight questions.
   (a) The first two problems are two pages, representing topics M1 and M2. You should do both of them, and they are **worth 30 points each**.
   (b) The remaining six problems represent topics S1 through S6. **You will be graded on your best four**, with a few possible bonus points if you also do well on the other two.
   (c) Doing four secondary topics well is much better than doing six poorly.
   (d) If you perform well on a question on this test it will update your mastery scores. Achieving a 18/20 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

M1a		M1b		M1c	
M2a		M2b		M2 c	
S1		S2		S3	
S4		S5		S6	
Σ					

Major Topic 1. Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Find the tangent line to $f(x) = \ln(\arctan(x) + 1) - 2$ at the point $x = 0$.

Solution:

We know that $f(0) = \ln(\arctan(0) + 1) - 2 = \ln(1) = -2$.

We have $f'(x) = \frac{1}{(1+x^2)(\arctan(x)+1)}$ and thus $f'(0) = \frac{1}{1\cdot(0+1)} = 1$. So the equation of the tangent line is

$$y + 2 = x$$

(b) Compute $\displaystyle\int \frac{1+e^x}{x+e^x}\,dx =$

Solution: We take $u = x + e^x$ so $du = 1 + e^x\,dx$. Then

$$\begin{aligned}\int_0^3 \frac{1+e^x}{x+e^x}\,dx &= \int_1^{3+e^3} \frac{1}{u}\,du \\ &= \ln|u| = \ln|x+e^x| + C.\end{aligned}$$

(c) $\displaystyle\int_0^{1/\sqrt[3]{2}} \frac{x^2}{\sqrt{1-x^6}}\,dx =$

Solution:

The obvious substitution is $u = 1 - x^6$ but that doesn't really work. After experimenting we see $u = x^3$ gives $du = 3x^2\,dx$ and thus

$$\begin{aligned}\int_0^{1/\sqrt[3]{3}} \frac{x^2}{\sqrt{1-x^6}}\,dx &= \int_0^{1/2} \frac{1}{3}\frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{3}\arcsin(u)\Big|_0^{1/2} \\ &= \frac{1}{3}\left(\arcsin(1/2) - \arcsin(0)\right) = \frac{1}{3}\left(\pi/6 - 0\right) = \frac{\pi}{18}.\end{aligned}$$

Alternatively we could compute that

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \frac{1}{3}\arcsin(x^3)$$

and thus

$$\begin{aligned}\int_0^{1/\sqrt[3]{2}} \frac{x^2}{\sqrt{1-x^6}}\,dx &= \frac{1}{3}\arcsin(x^3)\Big|_0^{1/\sqrt[3]{2}} \\ &= \frac{1}{3}\left(\arcsin(1/2) - \arcsin(0)\right) = \frac{1}{3}\left(\pi/6 - 0\right) = \frac{\pi}{18}.\end{aligned}$$

Major Topic 2 (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

Solution: Since we see $\sqrt{4-x^2}$ we want to try a trig substitution. So we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$. Then when $x = 0$ we have $\theta = 0$ and when $x = 1$ we have $\sin(\theta) = 1/2$ and thus

$\theta = \pi/6$. Then we get

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int_0^{\pi/6} \frac{4\sin^2(\theta)}{\sqrt{4-4\sin^2(\theta)}} \cdot 2\cos\theta\,d\theta \\
&= \int_0^{\pi/6} \frac{4\sin^2(\theta)}{2\sqrt{\cos^2(\theta)}} \cdot \cos(\theta)\,d\theta \\
&= \int_0^{\pi/6} 4\sin^2(\theta)\,d\theta = \int_0^{\pi/6} 2 - 2\cos(2\theta)\,d\theta \\
&= 2\theta - \sin(2\theta)\Big|_0^{\pi/6} = \Big(\pi/3 - \sin(\pi/3)\Big) - \Big(0 - \sin(0)\Big) \\
&= \pi/3 - \sqrt{3}/2.
\end{aligned}$$

(b) $\displaystyle\int x^2 e^{3x}\,dx$

Solution: We use integration by parts. Take $u = x^2, dv = e^{3x}\,dx$ so $du = 2x\,dx, v = e^{3x}/3$. Then

$$\begin{aligned}
\int x^2 e^{3x}\,dx &= x^2\frac{e^{3x}}{3} - \int \frac{2}{3}xe^{3x}\,dx \\
&= \frac{x^2e^{3x}}{3} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}\right) \\
&= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + C.
\end{aligned}$$

(c) $\displaystyle\int \frac{x-8}{(x+1)(x-2)^2}\,dx$

Solution: We use a partial fractions decomposition.

$$\begin{aligned}
\frac{x-8}{(x+1)(x-2)^2} &= \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \\
\frac{x-8}{(x-2)^2} &= A + \frac{B(x+1)}{x-2} + \frac{C(x+1)}{(x-2)^2} \\
\frac{x-8}{x+1} &= \frac{A(x-2)^2}{x+1} + B(x-2) + C
\end{aligned}$$

Plugging $x = -1$ into the middle equation gives $-1 = A$, and plugging $x = 2$ into the last equation gives $-2 = C$. So we just need to find B. But plugging in our values for A and C and setting $x = 0$ gives

$$\begin{aligned}
\frac{x-8}{(x+1)(x-2)^2} &= \frac{-1}{x+1} + \frac{B}{x-2} + \frac{-2}{(x-2)^2} \\
\frac{-8}{4} &= -1 + \frac{B}{-2} + \frac{-2}{4} \\
4 &= 2 + B + 1 \\
1 &= B.
\end{aligned}$$

Thus

$$\begin{aligned}
\int \frac{x-8}{(x+1)(x-2)^2}\,dx &= \int \frac{-1}{x+1} + \frac{1}{x-2} + \frac{-2}{(x-2)^2}\,dx \\
&= -\ln|x+1| + \ln|x-2| + \frac{2}{x-2} + C.
\end{aligned}$$

Secondary Topic 1. Let $f(x) = \sqrt{2e^x + e^{3x} + 1}$. Find $(f^{-1})'(2)$.

Solution: Plugging in numbers, we see that $f(0) = \sqrt{2+1+1} = 2$. Then by the Inverse Function Theorem we have $(f^{-1})'(2) = \frac{1}{f'(0)}$. But

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(2e^x + e^{3x} + 1\right)^{-1/2}\left(2e^x + 3e^{3x}\right) \\ &= \frac{2e^x + 3e^{3x}}{2\sqrt{2e^x + e^{3x} + 1}} \\ f'(0) &= \frac{2+3}{2\sqrt{2+1+1}} = \frac{5}{4}. \end{aligned}$$

Thus by the inverse function theorem we have

$$(f^{-1})'(2) = \frac{4}{5}.$$

Secondary Topic 2.

Find $\lim_{x\to+\infty}(e^x+1)^{1/x}$.

Solution: We have

$$\begin{aligned} y &= (e^x+1)^{1/x} \\ \ln(y) &= \frac{1}{x}\ln|e^x+1| = \frac{\ln(e^x+1)}{x}. \end{aligned}$$

Then

$$\begin{aligned} \lim_{x\to+\infty}\frac{\ln(e^x+1)^{\nearrow\infty}}{x_{\searrow\infty}} &=^{\text{L'H}} \lim_{x\to+\infty}\frac{\frac{e^x}{e^x+1}}{1} = \lim_{x\to+\infty}\frac{e^{x\nearrow\infty}}{e^x+1_{\searrow infty}} \\ &=^{\text{L'H}} \lim_{x\to+\infty}\frac{e^x}{e^x} = \lim_{x\to+\infty} 1. \end{aligned}$$

Thus $\lim_{x\to+\infty} y = e^1 = e$.

Secondary Topic 3.

Use the Trapezoid rule with six intervals to estimate $\int_{-4}^{2} x^2+1\,dx$.

Solution:

$$\begin{aligned} \int_{-4}^{2} x^2+1\,dx &\approx \frac{f(-4)+f(-3)}{2} + \frac{f(-3)+f(-2)}{2} + \frac{f(-2)+f(-1)}{2} \\ &\quad + \frac{f(-1)+f(0)}{2} + \frac{f(0)+f(1)}{2} + \frac{f(1)+f(2)}{2} \\ &= \frac{17+10}{2} + \frac{10+5}{2} + \frac{5+2}{2} + \frac{2+1}{2} + \frac{1+2}{2} + \frac{2+5}{2} \\ &= \frac{1}{2}(27+15+7+3+3+7) = \frac{62}{2} = 31. \end{aligned}$$

Alternatively, we could write

$$\begin{aligned} \int_{-4}^{2} x^2+1\,dx &\approx \frac{1}{2}\Big(f(-4)+2f(-3)+2f(-2)+2f(-1)+2f(0)+2f(1)+f(2)\Big) \\ &= \frac{1}{2}\Big(17+20+10+4+2+4+5\Big) = \frac{1}{2}\cdot 62 = 31. \end{aligned}$$

Secondary Topic 4.

Compute $\displaystyle\int_1^{10} \frac{1}{\sqrt[3]{x-2}}\,dx$.

Solution:

We must split the integral up into two parts:

$$
\begin{aligned}
\int_1^{10} \frac{1}{\sqrt[3]{x-2}}\,dx &= \int_1^{2} \frac{1}{\sqrt[3]{x-2}}\,dx + \int_2^{10} \frac{1}{\sqrt[3]{x-2}}\,dx \\
&= \lim_{s\to 2^-} \int_1^{s} \frac{dx}{\sqrt[3]{x-2}} + \lim_{t\to 2^+} \int_t^{10} \frac{dx}{\sqrt[3]{x-2}} \\
&= \lim_{s\to 2^-} \frac{3}{2}(x-2)^{2/3}\Big|_1^s + \lim_{t\to 2^+} \frac{3}{2}(x-2)^{2/3}\Big|_t^{10} \\
&= \left(\lim_{s\to 2^-} \frac{3(s-2)^{2/3}}{2} - \frac{3}{2}\right) + \left(\lim_{t\to 2^+} \frac{3\cdot 8^{2/3}}{2} - \frac{3(t-2)^{2/3}}{2}\right) \\
&= \frac{3}{2}\cdot 0 - \frac{3}{2} + \frac{12}{2} - \frac{3}{2}\cdot 0 = \frac{9}{2}.
\end{aligned}
$$

Secondary Topic 5.

Compute the arc length of the curve $y = \frac{1}{3}(x^2-2)^{3/2}$ as x varies from 3 to 6.

Solution: We have $y' = \frac{1}{3}\cdot\frac{3}{2}(x^2-2)^{1/2}\cdot 2x = x\sqrt{x^2-2}$. So the arc length is

$$
\begin{aligned}
L &= \int_3^6 \sqrt{1+y'^2}\,dx = \int_3^6 \sqrt{1+x^2(x^2-2)}\,dx \\
&= \int_3^6 \sqrt{1-2x^2+x^4}\,dx = \int_3^6 \sqrt{(x^2-1)^2}\,dx \\
&= \int_3^6 x^2-1\,dx = \frac{x^3}{3} - x\Big|_3^6 \\
&= \Big(72-6\Big) - \Big(9-3\Big) = 60.
\end{aligned}
$$

Secondary Topic 6. Find a (specific) solution to the initial value problem $y' = (y^2+1)(2x+3)$ if $y(0) = \sqrt{3}$.

Solution:

$$
\begin{aligned}
\frac{dy}{1+y^2} &= 2x+3\,dx \\
\arctan(y)x^2 &+ 3x + C \\
y &= \tan(x^2+3x+C) \\
\sqrt{3} &= \tan(C) \qquad \Rightarrow \qquad C = \pi/3 \\
y &= \tan(x^2+3x+\pi/3).
\end{aligned}
$$