Math 1231 Practice Midterm Solutions

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1. These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
2. You will have 75 minutes for this test.
3. You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
4. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.
5. This test has eight questions, over five pages. **You should not answer all eight questions.**
   (a) The first two problems are two pages, representing topics M1 and M2. You should do both of them, and they are worth 20 points each.
   (b) The remaining six problems represent topics S1 through S6. You will be graded on your best four, with a few possible bonus points if you also do well on the other two.
   (c) Doing four secondary topics well is much better than doing six poorly.
   (d) If you perform well on a question on this test it will update your mastery scores. Achieving a 18/20 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

1		5	
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Problem 1 (M1)**.** Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Find the tangent line to $h(x) = \arcsin(e^x)$ at $\ln(1/2)$.

Solution: We have $h'(x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$, so $h'(\ln(1/2)) = \frac{e^{\ln 1/2}}{\sqrt{1-e^{2\ln(1/2)}}} = \frac{1/2}{\sqrt{1-1/4}} = \frac{1}{\sqrt{3}}$. We also have $h(\ln(1/2)) = \arcsin(1/2) = \pi/6$.

Thus the equation of the tangent line is

$$y - \pi/6 = \frac{1}{\sqrt{3}}(x - \ln(1/2)).$$

(b) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx =$

Solution: We take $u = 1/x$ so $du = \frac{-1}{x^2}\,dx$. Then

$$\begin{aligned}\int_1^2 \frac{e^{1/x}}{x^2}\,dx &= \int_1^{1/2} -e^u\,du \\ &= -e^u\big|_1^{1/2} = -e^{1/2} + e^1 = e - \sqrt{e}.\end{aligned}$$

(c) $\displaystyle\int \frac{\cos(x)\sin(x)}{1+\cos^4(x)}\,dx =$

Solution:

We can take $u = \cos(x)$ so that $du = -\sin(x)\,dx$. Then

$$\int \frac{\cos(x)\sin(x)}{1+\cos^4(x)}\,dx = \int \frac{-u}{1+u^4}\,du$$

Then we can set $v = u^2$ so that $dv = 2u\,du$ and we get

$$\begin{aligned}\int \frac{-u}{1+u^4}\,du &= \int \frac{-1}{2}\frac{1}{1+v^2}\,dv = \frac{-1}{2}\arctan(v) + C \\ &= \frac{-1}{2}\arctan(u^2) + C = \frac{-1}{2}\arctan(\cos^2(x)) + C.\end{aligned}$$

Problem 2 (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) $\displaystyle\int \frac{2x+1}{\sqrt{x^2-1}}\,dx$

Solution: Since we see $\sqrt{x^2-1}$ we want to try a trig substitution. (You might try $u = x^2 - 1$ first, which almost works, but doesn't quite). So we set $x = \sec\theta$ and $dx = \sec\theta\tan\theta\,d\theta$. We have

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x^2-1}}\,dx &= \int \frac{2\sec\theta+1}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta \\ &= \int \frac{2\sec^2\theta\tan\theta + \sec\theta\tan\theta}{\tan\theta}\,d\theta \\ &= \int 2\sec^2\theta + \sec\theta\,d\theta \\ &= 2\tan\theta + \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

If $\sec\theta = x$ then θ is in a triangle with hypotenuse x and adjacent side 1 and thus opposite side $\sqrt{x^2-1}$. Thus $\tan\theta = \sqrt{x^2-1}$. This is good, since this formula appeared in our original question, and we see that

$$\int \frac{2x+1}{\sqrt{x^2-1}}\,dx = 2\sqrt{x^2-1} + \ln\left|x + \sqrt{x^2-1}\right| + C.$$

(b) $\int x \sec^2 x \, dx$

Solution: We use integration by parts. Take $u = x, dv = \sec^2 x \, dx$ so $du = dx, v = \tan x$. Then

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln|\cos x| + C.$$

(c) $\int_0^1 \frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)} \, dx$

Solution: We use a partial fractions decomposition.

$$\frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - x - 1}$$
$$3x^2 - 6x + 1 = A(x^2 - x - 1) + (Bx + C)(x - 2).$$

Plugging in $x = 2$ gives us that $1 = A$. Plugging in $x = 0$ gives $1 = -A - 2C = -1 - 2C$ and thus $C = -1$. Then plugging in $x = 1$ gives $-2 = -A - B - C = -1 - B + 1$ and thus $B = 2$. So we have

$$\begin{aligned}\int_0^1 \frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)} \, dx &= \int_0^1 \frac{1}{x-2} + \frac{2x - 1}{x^2 - x - 1} \, dx \\ &= \left(\ln|x - 2| + \ln|x^2 - x - 1|\right)\Big|_0^1 \\ &= \ln(1) + \ln(1) - \ln(2) - \ln(1) = -\ln(2).\end{aligned}$$

Problem 3 (S1)**.** Let $f(x) = \sqrt[3]{x^5 + x^4 + x^3 + x^2 + 2x}$. Find $(f^{-1})'(4)$.

Solution: Plugging in numbers, we see that $f(2) = \sqrt[3]{32 + 16 + 8 + 4 + 4} = \sqrt[3]{64} = 4$. Then by the Inverse Function Theorem we have $(f^{-1})'(4) = \frac{1}{f'(2)}$. But

$$f'(x) = \frac{1}{3}\left(x^5 + x^4 + x^3 + x^2 + 2x\right)^{-2/3}\left(5x^4 + 4x^3 + 3x^2 + 2x + 2\right)$$
$$f'(2) = \frac{1}{3}(64)^{-2/3}(80 + 32 + 12 + 4 + 2) = \frac{130}{48} = \frac{65}{24}.$$

Thus by the inverse function theorem we have

$$(f^{-1})'(4) = \frac{24}{65}.$$

Problem 4 (S2)**.**

Find $\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)}$.

Solution: $\lim_{x\to 0} 2\sin(x) - \sin(2x) = 0 - 0 = 0$, and $\lim_{x\to 0} x - \sin(x) = 0$, so we can use L'Hospital's Rule.

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} &=^{\text{L'H}} \lim_{x\to 0} \frac{2\cos(x) - 2\cos(2x)^{\nearrow 0}}{1 - \cos(x)_{\searrow 0}} \\ &=^{\text{L'H}} \lim_{x\to 0} \frac{-2\sin(x) + 4\sin(2x)^{\nearrow 0}}{\sin(x)_{\searrow 0}} \\ &=^{\text{L'H}} \lim_{x\to 0} \frac{-2\cos(x) + 8\cos(2x)}{\cos(x)} = \frac{6}{1} = 6.\end{aligned}$$

Problem 5 (S3)**.**

Use Simpson's rule and six intervals to estimate $\int_0^6 x^4\,dx$.

Solution:

$$\begin{aligned}\int_0^6 x^4\,dx &\approx \frac{1}{3}\left(0^4+4\cdot 1^4+2\cdot 2^4+4\cdot 3^4+2\cdot 4^4+4\cdot 5^4+6^4\right)\\ &= \frac{1}{3}\left(0+4+32+324+512+2500+1296\right) = \frac{4668}{3} = 1556.\end{aligned}$$

Problem 6 (S4)**.**

Does $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ converge or diverge? Why?

Solution:

We split the integral up into two parts:

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx.$$

The first integral is a finite integral of a continuous function and thus converges. Then for $x \geq 1$ we have $\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2}$. From class we know that $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, so $\int_1^{+\infty} \frac{x}{x^3+1}\,dx$ also converges. Thus the original integral is convergent.

Problem 7 (S5)**.**

Find the surface area of the surface obtained by rotating $y = \sqrt{5+4x}$ for $-1 \leq x \leq 1$ about the x-axis.

Solution: We have $y' = \frac{1}{2}(5+4x)^{-1/2}\cdot 4 = \frac{2}{\sqrt{5+4x}}$, so $ds = \sqrt{1+\frac{4}{5+4x}}dx$. Then

$$\begin{aligned}A &= \int_{-1}^1 2\pi y\,ds = 2\pi\int_{-1}^1 \sqrt{5+4x}\sqrt{1+\frac{4}{5+4x}}dx\\ &= 2\pi\int_{-1}^1 \sqrt{5+4x+4}\,dx = 2\pi\int_{-1}^1 \sqrt{9+4x}\,dx\\ &= 2\pi\left(\frac{2}{3}(9+4x)^{3/2}\cdot\frac{1}{4}\right)\Bigg|_{-1}^1 = 2\pi\left(\frac{1}{6}13\sqrt{13}-\frac{1}{6}5\sqrt{5}\right) = \frac{\pi}{3}\left(13\sqrt{13}-5\sqrt{5}\right).\end{aligned}$$

Problem 8 (S6)**.** Find a (specific) solution to the initial value problem $y'/x - y = 1$ if $y(0) = 3$

Solution:

$$\begin{aligned}y'/x &= 1+y\\ \frac{dy}{1+y} &= x\,dx\\ \ln|1+y|&x^2/2+C\\ 1+y &= e^{x^2/2}e^C\\ y &= Ke^{x^2/2}-1\\ 3 &= K-1 \Rightarrow K=4\\ y &= 4e^{x^2/2}-1.\end{aligned}$$