Math 322 Fall 2016
Number Theory HW 9
Due Friday, November 11

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem.

($\star$) **Starred Problem:** Let b be an inverse of a modulo n. Prove that $\operatorname{ord}_n a = \operatorname{ord}_n b$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. **Definition 1.** If f, g are arithmetic functions, we define the *Dirichlet convolution* of f and g to be

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

We define the function $\iota(n)$ by

$$\iota(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

We say g is the inverse of f (under Dirichlet convolution) if $f * g = \iota$.

   (a) Show that $f * g = g * f$; that is, Dirichlet convolution is commutative.

   (b) Show that $\iota(n)$ is multiplicative.

   (c) Show that $\iota * f = f$ for any arithmetic function f.

2. Prove that if f and g are multiplicative functions, then so is $f * g$.

3. Prove that if n is a natural number, then $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

4.

Definition 2. The *von Mangoldt function* Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & n = p^k \text{ for } p \text{ prime, } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\sum_{d|n} \Lambda(d) = \log n$ for any natural number n.
(Hint: recall $\log(a^k b^\ell) = k \log(a) + b \log(\ell)$).

(b) Use Möbius inversion to show that

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

(Hint: treat the case $n = 1$ separately).

5. Compute

(a) $\mu(12)$
(b) $\mu(210)$
(c) $\text{ord}_{11} 3$
(d) $\text{ord}_{10} 7$

6. (a) Show that 5 is a primitive root of 6.
(b) Show that 12 has no primitive roots.