1 Limits

1.1 What is a limit?

Poll Question 1.1.1. Suppose $|f(x)| \leq 3$ and $|g(x)| \leq x^2$. What can we say about $|f(x)+g(x)|$ and $|f(x)-g(x)|$?

$|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq 3+x^2$ (by the triangle inequality).

$|f(x)-g(x)| \geq |f(x)|-|g(x) \geq |f(x)|-x^2$ (by the reverse triangle inequality). We can't say anything more about the $|f(x)|$ bit because knowing that $|f(x)|$ is *smaller* than something doesn't tell us anything about what it's *bigger* than.

Poll Question 1.1.2. What value "should" the function given in the following graph have at $x=1$?

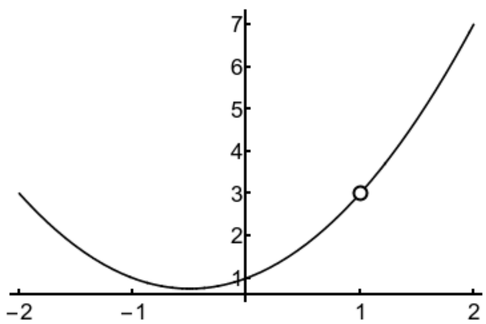

Figure 1.1: The graph of $\frac{x^3-1}{x-1}$

In this section we will study the idea of "limits". Recall a function takes an input and gives an output. The core idea of a limit is to look at the outputs for inputs "near" a given input, in order to answer the question, essentially, of what an output "should be" when we don't have a good one..

most of you have seen an informal characterization of limits before

Definition 1.1 (informal)**.** Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x\to a} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.

Graphically, this means that if the x coordinate is near a then the y coordinate is near L. Pictorially, if you draw a small enough circle around the point $(a,0)$ on the x-axis and look at the points of the graph above and below it, you can force all those points to be close to L.

Remark 1.2. We specifically do not consider the value of f at a when talking about limits. Limits were invented to deal with times when either a isn't in the domain of f, or when $f(a) \neq \lim_{x\to a} f(x)$.

Example 1.3.

1. If $f(x) = 3x$ then $\lim_{x\to 1} f(x) = 3$.
2. If $f(x) = x^2$ then $\lim_{x\to 0} f(x) = 0$.
3. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x\to 1} f(x) = 2$.

We'd like to take this definition and translate it into mathematical language, making it more precise at the same time.

Definition 1.4. Suppose a is a real number, and f is a function defined on some open interval containing a, except possibly for at a. We say the *limit* of $f(x)$ as x approaches a is L, and write

$$\lim_{x\to a} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.

Importantly, you should notice that this is *exactly the same thing we said before!* ϵ represents "how close we want $f(x)$ to get to L" and δ represents "how close x needs to get to a".

Then this definition says that if we pick any margin of error $\epsilon > 0$, then there is some distance δ such that if x is within distance δ of a, then $f(x)$ is within our margin of error ϵ of L.

Remark 1.5. The Greek letter epsilon (ϵ) became the letter "e", and stands for "error". The Greek letter delta (δ) became the letter "d", and stands for "distance". This isn't just a mnemonic for you; this is actually why those letters were chosen.

Example 1.6.

1. If $f(x) = 3x$ then prove $\lim_{x\to 1} f(x) = 3$.

   Let $\epsilon > 0$ and set $\delta = \underline{\epsilon/3}$. Then if $|x-1| < \delta$ then

$$|f(x) - 3| = |3x - 3| = 3|x-1| < 3\delta = \epsilon.$$

2. If $f(x) = x^2$ then prove $\lim_{x\to 0} f(x) = 0$.

   Let $\epsilon > 0$ and set $\delta = \underline{\sqrt{\epsilon}}$. Then if $|x - 0| < \delta$, then
$$|f(x) - 0| = |x^2| = |x|^2 < (\sqrt{\epsilon})^2 = \epsilon.$$

3. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x\to 1} f(x) = 2$.

   This is harder to see at first, until we recall or notice that this function is mostly the same as $x + 1$.

   Let $\epsilon > 0$ and let $\delta = \underline{\epsilon}$. Then if $0 < |x - 1| < \delta$, we have
$$\begin{aligned} |f(x) - 2| &= \left|\frac{x^2-1}{x-1} - 2\right| \\ &= |x + 1 - 2| && \text{since } x \neq 1 \\ &= |x - 1| < \delta = \epsilon. \end{aligned}$$

Remark 1.7. Despite the fact that we set δ as the first thing we do in the proof, we often figure out what it should be last. I strongly recommend beginning your proof by writing "And set $\delta =$ ____" and then working out the proof. By the time you get to the end you'll know what δ needs to be and you can go back and fill in th blank.

Poll Question 1.1.3. If $f(x) = 4x - 2$ then find (with proof!) $\lim_{x\to -2} f(x)$.

We first need to generate a "guess". This is a nice function, so it seems like the answer should be close to $f(-2) = -10$.

Let $\epsilon > 0$ and set $\delta = \underline{\epsilon/4}$. Then if $|x - (-2)| < \delta$ we compute
$$|f(x) + 10| = |4x - 2 + 10| = |4x + 8| = 4|x + 2| < 4\delta = \epsilon.$$

Example 1.8. If $f(x) = x^2$ find (with proof!) $\lim_{x\to 3} f(x)$.

We first need to generate a "guess". This is a nice function, so it seems like the answer should be close to $f(3) = 9$.

Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/7, 1}$. Then if $|x - 3| < \delta$ we compute
$$|x^2 - 9| = |x + 3| \cdot |x - 3| < |x + 3|\delta$$
but this is kind of a problem because we still have an x floating around. But logically, we know that if δ is small enough, x will be close to 3 and thus $|x + 3|$ will be close to 6.

To guarantee that $|x+3|$ is actually close to 6, we'll require $\delta \leq 1$ as well. Then we compute

$$\begin{aligned} |x^2-9| &< |x+3|\delta = |(x-3)+6| \cdot \delta \\ &\leq (|x-3|+|6|)\,\delta && \text{by the triangle inequality} \\ &< (1+6)\delta = 7\delta. \end{aligned}$$

Notice we said that $|x+3|$ would be close to 6, and what we actually showed is that $|x+3| \leq 7$–which of course it is if it is close to 6.

So now we just need to make sure δ is small enough that $7\delta \leq \epsilon$, so in addition to letting $\delta \leq 1$ we also let $\delta \leq \epsilon/7$, so we have

$$|x^2-9| < 7\delta = 7\epsilon/7 = \epsilon.$$

Remark 1.9.

- We often use an approach of isolating all our xs and turning them into an $x-3$ or $x-a$ or whatever we *know how to control*. Since in example 1.8 we know that $|x-3| < \delta$ we want to turn all our xs into $|x-3|$s. Then we can deal with whatever is left over.
- Notice that here we didn't actually say what δ is; we just listed some properties it needs to have, by saying that $\delta \leq \epsilon/12, 1$. If we want to pick out a specific number, we can write $\delta = \min(\epsilon/12, 1)$, but this isn't actually necessary.

Example 1.10. If $f(x) = x^2 + x$, find (with proof) $\lim_{x\to 2} f(x)$.

This is a nice function, so it seems like the answer should be close to $f(2) = 6$.

Let $\epsilon > 0$ and set $\delta < \underline{\sqrt{\epsilon/2}, \epsilon/10}$. Then if $|x-2| < \delta$ we have

$$\begin{aligned} |f(x)-6| &= |x^2+x-6| = |(x^2-4)+(x-2)| \\ &\leq |x^2-4| + |x-2| && \text{(triangle inequality)} \\ &= |x-2| \cdot |x+2| + |x-2| = |x-2|\,(|x+2|+1) \\ &= |x-2|\,(|x-2+4|+1) \leq |x-2|\,(|x-2|+5) && \text{(triangle inequality)} \\ &< \delta(\delta+5) = \delta^2 + 5\delta. \end{aligned}$$

You could try to figure out exactly when $\delta^2 + 5\delta = \epsilon$, and after some quadratic formula-ing you'd find you need $\delta \leq \frac{-5+\sqrt{25+4\epsilon}}{2}$. But that's tedious and actually way too much work. (But if you prefer this approach it's perfectly acceptable).

It's easier to instead list two conditions: we let $\delta \leq \sqrt{\epsilon/2}, \epsilon/10$. Then $\delta^2 \leq \epsilon/2$ and $5\delta \leq \epsilon/2$, and we have

$$|f(x)-6| < \delta^2 + 5\delta \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Example 1.11. If $f(x) = 1/x$, find (with proof) $\lim_{x\to 4} f(x)$.

Since $f(x)$ is a nice function, we guess $f(4) = 1/4$.

Let $\epsilon > 0$ and set $\delta < \underline{1, 12\epsilon}$. Then we compute

$$|f(x) - 1/4| = |1/x - 1/4| = \left|\frac{4-x}{4x}\right| = \frac{|x-4|}{|4x|} < \frac{\delta}{|4x|}$$

and we need to do something about the x on the bottom. In this case we need to ensure that $|4x|$ is *big* enough since we're dividing by it. We see that $|4x| = |4(x-4+4)| = |4(x-4)+16|$; how can we make this bigger than something?

Here we use the inverse triangle inequality, after a bit of rewriting. We compute

$$\begin{aligned}
|4(x-4) + 16| = |16 - 4(4-x)| \geq |16| - |4(4-x)| &= 16 - 4|x-4| \\
|x-4| &< \delta < 1 \\
-4|x-4| &> -4 \\
16 - 4|x-4| &> 12.
\end{aligned}$$

Now we can compute

$$|1/x - 1/4| < \frac{\delta}{|4x|} < \frac{12\epsilon}{12} = \epsilon.$$

Example 1.12. If $f(x) = \frac{x-1}{x^2-1}$ then find (with proof!) $\lim_{x\to 1} f(x)$?

We notice that if $x \neq 1$, then $f(x) = \frac{1}{x+1}$, and so we guess $\lim_{x\to 1} f(x) = 1/2$.

Let $\epsilon > 0$ and let $\delta = \underline{\epsilon, 1}$. Then if $|x-1| < \delta$ we have

$$\begin{aligned}
|f(x) - 1/2| &= \left|\frac{x-1}{x^2-1} - 1/2\right| = \left|\frac{1}{x+1} - 1/2\right| \qquad \text{because } x \neq -1 \\
&= \left|\frac{2}{2x+2} - \frac{x+1}{2x+2}\right| = \left|\frac{1-x}{2x+2}\right| \\
&= \frac{|x-1|}{|2(x-1)+4|}.
\end{aligned}$$

We want to make the top small, so require $\delta < \epsilon$. We want the bottom to be big, say we want it to be at least two. We see that

$$|2(x-1) + 4| = |4 - 2(1-x)| \geq 4 - 2|1-x| > 4 - 2\delta$$

so if we require $\delta < 1$ this gives us

$$|2(x-1) + 4| > 2.$$

Thus we have

$$|f(x) - 1/2| = \frac{|x-1|}{|2x+2|} < \frac{\delta}{4-2\delta} < \delta/2 < \epsilon/2 < \epsilon.$$

Example 1.13. Now suppose

$$f(x) = \begin{cases} \frac{x-1}{x^2-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

What is $\lim_{x\to 1} f(x)$?

This looks really nasty, but is actually easy after we already did Example 1.12.

The limit doesn't care about what happens at any one specific point, and especially doesn't care about what happens at 1. So for our purposes, this function is the same as $f(x) = \frac{x-1}{x^2-1}$, and thus the limit is, as before, 1/2.

Let $\epsilon > 0$, and let $\delta < \epsilon, 1$. Then

$$|f(x) - 1/2| = \left| \frac{x-1}{x^2-1} - 1/2 \right| < \epsilon$$

as computed in Example 1.12. (This is a completely valid proof as written!)

1.2 Limits that Don't Exist

In the last section we proved that a bunch of limits exist. Now we'll look at some functions and limits that don't behave so nicely.

Example 1.14. The *Heaviside Function* or *step function* is given by

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

What is $\lim_{x\to 0} H(x)$?

Looking at the graph it seems like no limit exists; when x is close to zero, sometimes $H(x)$ is 0 and sometimes $H(x)$ is 1, and you can't get "close enough" to make that stop.

We want to prove that no limit exists, so we have to look at our definition–and do the exact opposite of what we do to prove a limit does exist. Normally we want to say that for *any* expected error, we can get close enough to be within that error. So we say we can start with any ϵ, and then find a good enough δ.

In this situation we want to say there is some error we *cannot* hit. So we start by picking some ϵ, and then proving that no δ will work.

(I like to think of this as a game. When I say a limit exists, I'm telling you you can pick *any* ϵ, and I can find a δ. In contrast, here I'm saying that you have a winning move–there's some ϵ you can pick where I can't find a δ).

Proof. Suppose the limit exists–that is there is some number L such that for ever $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - 0| < \delta$ then $|H(x) - L| < \epsilon$.

Fix $\epsilon = 1/2$. (We will see why I picked this specific value in a bit). Then let $\delta > 0$ be any (positive) real number.

Let $x_1 = \delta/2$. Then $|x_1 - 0| = \delta/2 < \delta$. Thus by definition of limit,

$$\epsilon > |H(x_1) - L| = |1 - L|.$$

Thus $|1 - L| < 1/2$.

Now let $x_2 = -\delta/2$. Then $|x_2 - 0| = \delta/2 < \delta$, and by definition of limit,

$$\epsilon > |H(x_2) - L| = |0 - L| = |L|.$$

Thus $|L| < 1/2$.

So what does this mean? Since $|1 - L| < 1/2$ we know that L is within $1/2$ of 1. Since $|L| < 1/2$ we know that L is within $1/2$ of 0. But there are no numbers that are within $1/2$ of both 0 and 1, so L cannot exist!

We can translate this into more mathematical language in two different ways.

We can add our two inequalities, to get $|1 - L| + |L| < 1/2 + 1/2$. But if we look at the left hand side of that, that looks like part of the triangle inequality. So we have

$$1 > |1 - L| + |L| \geq |1 - L + L| = |1|$$

but this is false and thus we have a contradiction. So no such L can exist, and the limit does not exist.

Alternatively, we can use the inverse triangle inequality, which tells us that $1/2 > |1-L| \geq 1 - |L|$. Adding our two inequalities together now gives $1 - |L| + |L| < 1/2 + 1/2$ and thus $1 < 1$, which is a contradiction. So no such L exists, and the limit does not exist.

(Notice that all three of these arguments are essentially the same!)

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Example 1.15. What is $\lim_{x \to 1} H(x)$?

There's nothing funny going on here, it looks like the limit should be 1. And indeed proving this in this case is quite easy.

Let $\epsilon > 0$ and let $\delta = \underline{1}$. Then if $|x - 1| < \delta$ then in particular we have $|x - 1| < 1$ and thus $x - 1 > -1$ so $x > 0$ and $H(x) = 1$. Then we have$|H(x) - 1| = 0 < \epsilon$.

Example 1.16. Let

$$f(x) = \begin{cases} x & x < 1 \\ x+2 & x > 1 \end{cases}$$

What is $\lim_{x \to 3} f(x)$?

We guess 5. Let $\epsilon > 0$ and set $\delta \leq \underline{2, \epsilon}$. Then if $0 < |x-3| < \delta$, then we see that in particular $|x-3| < 2$. This implies that $x - 3 > -2$ and thus $x > 1$, so $f(x) = x+2$. Then

$$|f(x) - 5| = |x + 2 - 5| = |x - 3| < \delta \leq \epsilon.$$

Thus $\lim_{x \to 3} f(x) = 5$.

Example 1.17. Now show that $\lim_{x \to 1} f(x)$ does not exist.

Suppose (for contradiction) that $\lim_{x \to 1} f(x) = L$. Then set $\epsilon =$ and let $\delta > 0$. Pick $x_1 = 1 + \delta/2$ and $x_2 = 1 - \delta/2$.

Then $|x_1 - 1| = \delta/2 < \delta$, so we know that

$$\epsilon > |f(x_1) - L| = |x_1 + 2 - L| = |3 + \delta/2 - L|.$$

Similarly, $|x_2 - 1| = \delta/2 < \delta$ so we know that

$$\epsilon > |f(x_2) - L| = |x_2 - L| = |1 - \delta/2 - L| = |L + \delta/2 - 1|.$$

Adding these two equations gives

$$\begin{aligned} 2\epsilon &> |L + \delta/2 - 1| + |3 + \delta/2 - L| \\ &\geq |L + \delta/2 - 1 + 3 + \delta/2 - L| = |2 + \delta| \\ &\geq 2 + |\delta| > 2. \end{aligned}$$

But since $\epsilon = 1$ this gives us $2 > 2$ which is a contradiction. Thus no such limit exists.

Example 1.18. What is $\lim_{x \to 0} \sin(1/x)$?

If we look at a graph of the function, it's hard to see what the limit could be–the function jumps up and down crazily near 0, and it doesn't look like you can get "close enough" to avoid this.

Suppose there is some L with $\lim_{x \to 0} f(x) = L$. Let $\epsilon = \underline{1}$ and fix some $\delta > 0$. Write $f(x) = \sin(1/x)$. I claim there is some positive integer n such that $0 < \frac{2}{n\pi} < \delta$ (pick a number such that $n > \frac{2}{\pi\delta}$.) In fact, we can pick n so that $\sin(n\pi/2) = 1$.

Let $x_1 = \frac{2}{n\pi}$. Then $|x_1 - 0| < \delta$ by construction, and

$$|f(x_1) - L| = |\sin(n\pi/2) - L| = |1 - L| < \epsilon = 1.$$

Let $x_2 = \frac{-2}{n\pi}$. Then $|x_2 - 0| < \delta$, and

$$|f(x_2) - L| = |\sin(-n\pi/2) - L| = |-1 - L| < \epsilon = 1.$$

Informally: L must be within 1 of both the number 1 and the number -1; no number satisfies these conditions, so the limit cannot exist.

Formally: there are two ways to see this. One is to notice that $|1 + L| + |1 - L|$ looks like it comes from the triangle inequality. So we have

$$1 + 1 > |1 + L| + |1 - L| \geq |1 + L + 1 - L| = 2$$

which is a contradiction. Thus no such L exists.

The other way is to decide things will be easier if we can make all the Ls be in the same form. We have a $|1 - L| = |L - 1|$ and we can rewrite $|1 + L| = |L + 2 - 1| \leq |L - 1| + 2$. Now we can add our two inequalities together, and we get

$$2 = 1 + 1 > |1 - L| + |1 + L| \geq |1 - L| + |1 - L| + 2 \geq 2$$

which is impossible. Thus no such L exists.

A summary of the layout of these proofs: Assume a limit $\lim_{x\to a} f(x)$ exists. Pick a value for ϵ (you can leave this blank until later), and then let $\delta > 0$ be some real number.

Now find two points that are both close to your limit point a, but give very different outputs. Use the assumption that the limit exists to see that $\epsilon > |f(x_1) - L|$ and $\epsilon > |f(x_2) - L|$ and add these together; use the triangle inequality to cancel the Ls, and get $2\epsilon > f(x_1) - f(x_2)$.

Hopefully the right hand side is a constant (or bigger than a constant), and we can pick ϵ to be small enough that this can't work.

1.3 One-sided limits

We just proved that two different limits don't exist; but one of them is much, much nicer than the other. Looking at the graph, there are two plausible values you could give for $\lim_{x\to 0} H(x)$; for $\lim_{x\to 0} \sin(1/x)$ there are infinitely many. We can capture this difference with the idea of a *one-sided limit*:

Definition 1.19. Suppose a is a real number, and f is a function defined on some open interval $(a - h, a)$ We say the *limit* of $f(x)$ as x approaches a *from the left* is L, and write

$$\lim_{x\to a^-} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $a - \delta < x < a$ then $|f(x) - L| < \epsilon$.

Remark 1.20. This says that x has to be within δ of a, but also smaller than a–thus "on the left." This captures the idea that $f(x)$ gets close to L when x is sufficiently close to a but still smaller and to the left.

Thus this intuitively, this captures the idea that the outputs of a function get close to one value on one side of a point, and perhaps get close to a different value (or are simply ill-behaved) on the other side.

Notice the subscript a^- in the limit sign. Recall that we use a $-$ sign because we're looking at what happens for inputs less than a.

Remark 1.21. The $a - \delta < x < a$ might look unrelated to what we've done so far. But note that $|x - a| < \delta$ is the same as $-\delta < x - a < \delta$, which is the same as $a - \delta < x < a + \delta$. So this is just the left half our earlier $|x - a| < \delta$ requirement.

Of course we can make a similar definition for a one-sided limit from the right.

Definition 1.22. Suppose a is a real number, and f is a function defined on some open interval $(a, a + h)$ We say the *limit* of $f(x)$ as x approaches *a from the right* is L, and write

$$\lim_{x \to a^+} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $a < x < a + \delta$ then $|f(x) - L| < \epsilon$.

Remark 1.23. Notice that $|x - a| < \delta$ is the same as saying $-\delta < x - a < \delta$, which is the same as $a - \delta < x < a + \delta$. This makes our two-sided limit definition look a lot more like the one-sided definition.

Example 1.24. Let's compute $\lim_{x \to 0^-} H(x)$. It looks like this limit should be 0, since $H(x) = 0$ whenever $x < 0$.

So let $\epsilon > 0$ and $\delta = \underline{1}$. Then if $0 - 1 < x < 0$, then $|H(x) - 0| = |0 - 0| = 0 < \epsilon$.

Notice that $\lim_{x \to 0^-} H(x) \neq H(0)$. We say that $H(x)$ is "not continuous at 0", a concept we will discuss more in section 4.

Example 1.25. Now let's compute $\lim_{x \to 0^-} H(x)$. It looks like this limit should be 1, since $H(x) = 1$ whenever $x > 0$.

So let $\epsilon > 0$ and $\delta = \underline{1}$. Then if $0 < x < 0 + 1$, then $|H(x) - 1| = |1 - 1| = 0 < \epsilon$.

Example 1.26. Let

$$f(x) = \begin{cases} x^2 + 3 & x < -1 \\ 3x^2 & x \geq -1 \end{cases}$$

Find $\lim_{x \to -1^+} f(x)$.

From the right this looks like $3x^2$ so we guess 3.

Let $\epsilon > 0$ and set $\delta \leq \underline{1, \epsilon/9}$. Then if $-1 < x < -1 + \delta$, we compute

$$\begin{aligned} |f(x) - 3| &= |3x^2 - 3| && \text{since } x > -1 \\ &= 3|x^2 - 1| = 3|x-1| \cdot |x+1| < 3\delta|x-1| \\ &= 3\delta|x + 1 + (-2)| \leq 3\delta(|x+1| + |2|) \leq 3\delta(3) = 9\epsilon/9 = \epsilon. \end{aligned}$$