2 Limits involving infinity

In this section we'll talk about limits that involve infinity. Some limits deal with infinity as an output, and others deal with it as an input (or both).

Remark 2.1. Recall that infinity is not a number. Sometimes while dealing with infinite limits we might make statements that appear to treat infinity as a number. But it's not safe to treat ∞ like a true number and we will be careful of this fact.

2.1 Infinite limits

Definition 2.2. We write

$$\lim_{x \to a} f(x) = +\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily large (and positive).

We write

$$\lim_{x \to a} f(x) = -\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily negative.

We write

$$\lim_{x \to a} f(x) = \pm\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily positive or negative. We usually use this when both occur.

Formally:

Definition 2.3. We write

$$\lim_{x \to a} f(x) = +\infty$$

if for every $N > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > N$.

We write

$$\lim_{x \to a} f(x) = -\infty$$

if for every $N > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) < -N$.

We write

$$\lim_{x \to a} f(x) = \pm\infty$$

if for every $N > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x)| > N$.

Remark 2.4. Important note: If the limit of a function is infinity, the limit *does not exist*. This is utterly terrible English but I didn't make it up so I can't fix it. Most of the theorems that say "If a limit exists" are not including cases where the limit is infinite.

Also notice that if a limit is $+\infty$ or is $-\infty$, then it is also true to say that the limit is $\pm\infty$. Remember that ∞ isn't a number; the statement that a limit is $+\infty$ and that it's $\pm\infty$ are not mutually contradictory, and in fact the first implies the second. And both statements imply that the limit does not exist.

Thus it would sometimes be correct to say that $\lim_{x\to a} f(x) = +\infty$, or to say that $\lim_{x\to a} f(x) = \pm\infty$, or to say that $\lim_{x\to a} f(x)$ does not exist, all about the same function at the same time.

On a test you should always try to make the most precise statement you can. If a limit is $+\infty$ and you say that the limit does not exist, you are *correct* but you will not get full credit.

Example 2.5. The most important example, that we do all our work in reference to, is that $\lim_{x\to 0} 1/x = \pm\infty$. In particular, if x is very small and positive our output will be very large and positive; if x is very small and negative then our output will be very large and negative.

Fix some $N > 0$, and set $\delta = \underline{1/N}$. Then if $0 < |x - 0| < \delta$ then

$$|1/x| = 1/|x| > 1/\delta = 1/(1/N) = N.$$

Example 2.6. $\lim_{x\to 0} 1/x^2 = +\infty$. We can see that the values will get very large because the limit of the top is a constant and the limit of the bottom is 0; but both the numerator and the denominator will always be positive, since $x^2 \geq 0$ (and of course $1 \geq 0$).

Formally: fix some $N > 0$ and set $\delta = \underline{1/\sqrt{N}}$. Then if $0 < |x - 0| < \delta$ then

$$1/x^2 > 1/\delta^2 = 1/(1/\sqrt{N})^2 = N.$$

Thus $\lim_{x\to 0} 1/x^2 = +\infty$.

Notice that we didn't use absolute values in this proof—this is the difference between a limit being $+\infty$ and a limit being $\pm\infty$.

Notice also that we easily *could* prove that $\lim_{x\to 0} 1/x^2 = \pm\infty$.

Example 2.7. $\lim_{x\to 1} \frac{x}{x-1} = \pm\infty$. We see that the top will approach 1 and the bottom will approach 0 so the limit is $\pm\infty$. The top is always positive near 0; the bottom is positive to the right and negative to the left, so the sign flips. Thus the answer is $\pm\infty$.

Formally: Let $N > 0$ and set $\delta \leq \underline{1/2, 1/2N}$. Then if $0 < |x-1| < \delta$, we compute

$$\begin{aligned} \left|\frac{x}{x-1}\right| &= \frac{|1-(1-x)|}{|x-1|} \geq \frac{1-|x-1|}{|x-1|} \\ &\geq \frac{1-\delta}{|x-1|} > \frac{1/2}{\delta} = \frac{1/1}{1/2N} = N. \end{aligned}$$

Thus $\lim_{x\to 1} \frac{x}{x-1} = \pm\infty$.

Note that we can't be more precise, since the outputs of f get both very positive and very negative in a small window around 1. If we wanted to prove this, we could compute the one-sided limits:

If $N > 0$, set $\delta = \underline{1/N}$, and then if $1 < x < 1+\delta$ then

$$\begin{aligned} \frac{x}{x-1} &= \geq \frac{1}{1+\delta-1} \\ &\geq \frac{1}{\delta} > \frac{1}{1/N} = N. \end{aligned}$$

Thus $\lim_{x\to 1^+} \frac{x}{x-1} = +\infty$.

Now we compute the limit for the left. Let$N > 0$, set $\delta \leq \underline{1/2, 1/2N}$ and then if $1-\delta < x < 1$ then we want to show that $\frac{x}{x-1} < -N$. This gets tricky since we're dealing with a bunch of negative numbers; we'll deal with it in pieces.

We want to show the top is (relatively) large, so we compute

$$x > 1-\delta > 1-1/2 = 1/2.$$

This is relatively straightforward since all the numbers are positive.

For the bottom we observe that $-\delta < x-1 < 0$ and we have two negative numbers. This gives

$$\begin{aligned} -\delta &< x-1 \\ \frac{1}{x-1} &< \frac{-1}{\delta} \\ \frac{x}{x-1} &< \frac{-1/2}{\delta} = \frac{-1}{2\delta}. \end{aligned}$$

Finally we set $\delta \leq 1/2N$ so we get

$$\begin{aligned} \frac{1}{\delta} &\geq 2N \\ \frac{-1}{2\delta} &\leq -N. \end{aligned}$$

This gives us the desired result.

An alternative approach is to start by multiplying everything by -1. Then we want to show that $\frac{x}{1-x} \geq N$, and we have $0 < 1 - x < \delta$. Since now everything is positive we have

$$\begin{aligned} \frac{x}{1-x} &> \frac{1-\delta}{1-x} \geq \frac{1/2}{1-x} \\ &> \frac{1/2}{\delta} \geq \frac{1/2}{1/2N} = N. \end{aligned}$$

And since the two one-sided limits aren't the same, the two-sided limit is $\pm\infty$ and not strictly one or the other.

Poll Question 2.1.1. Compute with proof $\lim_{x \to -3} \frac{x}{(x+3)^2}$.

Let $N > 0$ and set $\delta \leq 2,$. Then if $0 < |x+3| < \delta$, we know that $-\delta < x + 3 < \delta$ and thus $-3 - \delta < x < -3 + \delta$, so

$$\begin{aligned} \frac{x}{(x+3)^2} &< \frac{-3+\delta}{(x+3)^2} \leq \frac{-1}{(x+3)^2} \\ &< \frac{-1}{\delta^2} \leq \frac{-1}{(1/\sqrt{N})^2} \\ &= \frac{-1}{1/N} = -N. \end{aligned}$$

Thus $\lim_{x \to -3} \frac{x}{(x+3)^2} = -\infty$.

2.2 Limits at infinity

We now ask what happens to a function as the input gets arbitrarily large (or negative).

Definition 2.8. Let f be a function defined for (a, ∞) for some number a. We write

$$\lim_{x \to +\infty} f(x) = L$$

to indicate that when x is large enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x) - L| < \epsilon$.

We write

$$\lim_{x \to +\infty} f(x) = \pm\infty$$

to indicate that when x is large enough, $f(x)$ is also very large or very negative. Formally, if for every $N > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x)| > N$.

Let f be a function defined for $(-\infty, a)$ for some number a. We write

$$\lim_{x \to -\infty} f(x) = L$$

to indicate that when x is negative enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x < -M$ then $|f(x) - L| < \epsilon$.

We write

$$\lim_{x \to -\infty} f(x) = \pm\infty$$

to indicate that when x is negative enough, $f(x)$ is also very large or very negative. Formally, if for every $N > 0$ there is a $M > 0$ so that if $x < -M$ then $|f(x)| > N$.

If the limit at infinity is finite we say that the curve $y = f(x)$ has a *horizontal asymptote* at the line $y = L$. If the limit at infinity is infinite we will sometimes talk about a *slant asymptote*.

Example 2.9. The most important limits to calculate, as before, are the limits of x and $1/x$. It seems reasonable that $\lim_{x \to +\infty} x = +\infty$ and $\lim_{x \to -\infty} x = -\infty$.

Let $N > 0$ and set $M = \underline{N}$. Then if $x > M$ we have $x > M = N$, so $\lim_{x \to +\infty} x = +\infty$. Similarly, if $x < -M$ then $x < -M = -N$ so $\lim_{x \to -\infty} x = -\infty$.

For $f(x) = 1/x$, we expect the limit to be 0. So let $\epsilon > 0$ and set $M = \underline{1/\epsilon}$. Then if $x > M$, we have

$$|f(x) - 0| = 1/|x| < 1/M = 1/(1/\epsilon) = \epsilon.$$

Similarly, if $x < -M$ then

$$|f(x) - 0| = 1/|x| < 1/|-M| = 1/M = \epsilon.$$

Example 2.10. What is $\lim_{x \to +\infty} 1/x^2$?

We can compute this directly or indirectly. Directly: Let $\epsilon > 0$ and let $N = 1/\sqrt{\epsilon}$. Then if $x > N$ we have

$$\left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < \frac{1}{N^2} = \frac{1}{(1/\sqrt{\epsilon})^2} = \epsilon.$$

Thus $\lim_{x \to +\infty} 1/x^2 = 0$.

Example 2.11. What is $\lim_{x \to +\infty} \frac{3}{x+3}$?

We expect that this is zero. So let $\epsilon > 0$ and set $M \geq \underline{3/\epsilon - 3, 1}$. Then if $x > M$ then

$$\left| \frac{3}{x+3} - 0 \right| = \frac{3}{|x+3|} = \frac{3}{x+3} < \frac{3}{M+3} \leq \frac{3}{3/\epsilon - 3 + 3} = \frac{3}{3/\epsilon} = \epsilon.$$

Example 2.12. What is $\lim_{x \to -\infty} \frac{2}{x^2}$?

We expect that this is also zero. So let $\epsilon > 0$ and set $M \geq \underline{\sqrt{2/\epsilon}}$. Then if $x < -M$, we have

$$\left| \frac{2}{x^2} - 0 \right| = \frac{2}{x^2} < \frac{2}{M^2} \leq \frac{2}{2/\epsilon} = \epsilon.$$

Example 2.13. What is $\lim_{x \to +\infty} \frac{2x+3}{x+5}$?

We guess 2. Let $\epsilon > 0$ and set $M \geq \underline{7/\epsilon - 5, 1}$. Then if $x > M$ we have

$$\left| \frac{2x+3}{x+5} - 2 \right| = \left| \frac{2x+3}{x+5} - \frac{2x+10}{x+5} \right| = \left| \frac{-7}{x+5} \right|$$
$$= \frac{7}{x+5} \leq \frac{7}{7/\epsilon} = \epsilon.$$

Example 2.14. Does $\lim_{x \to -\infty} \frac{1}{\sqrt{x}}$ exist? Why?

It doesn't because $\frac{1}{\sqrt{x}}$ is not defined for negative numbers, so we can't even approach $-\infty$.

Example 2.15. What is $\lim_{x \to +\infty} \frac{x^2}{x-3}$? We guess $+\infty$.

Let $N > 0$ and set $M \geq \underline{4, N}$. Then if $x > M$ we have

$$\frac{x^2}{x-3} > \frac{x^2}{x} = x > M \geq N.$$

Example 2.16. What is $\lim_{x \to -\infty} \frac{x^2}{x-3}$? We guess $-\infty$. (It's negative because the top will be positive and the bottom will be negative).

This is another one of those problems with hidden negative signs, which will get confusing. Let $N > 0$ and set $M \geq \underline{3, 2N}$. Then if $x < -M$ we observe that since $x - 3 \geq 2x$ (since $x < -M \leq -3$) then $\frac{1}{x-3} \leq \frac{1}{2x}$, and then we have

$$\frac{x^2}{x-3} \leq \frac{x^2}{2x} = x/2 < -M/2 \leq -N.$$

Example 2.17. What is $\lim_{x \to +\infty} \sin(x)$?

This is exactly like $\lim_{x \to 0} \sin(1/x)$. The limit does not exist for the same reason; as x gets bigger and bigger, the function $\sin(x)$ continues to oscillate and does not approach one value, nor does it increase without bound.

	$x \to a$ δ is small	$x \to +\infty$ M is large	$x \to -\infty$ M is large
$\lim f(x) = L$ ϵ is small	Let $\epsilon > 0$ and set $\delta \leq \underline{\quad}$. Then if $0 < \|x - a\| < \delta$ then $\|f(x) - L\| < \epsilon$.	Let $\epsilon > 0$ and set $M \geq \underline{\quad}$. Then if $x > M$ then $\|f(x) - L\| < \epsilon$.	Let $\epsilon > 0$ and set $M \geq \underline{\quad}$. Then if $x < -M$ then $\|f(x) - L\| < \epsilon$.
$\lim f(x) = \pm\infty$ N is large	Let $N > 0$ and set $\delta \leq \underline{\quad}$. Then if $0 < \|x - a\| < \delta$ then $\|f(x)\| > N$.	Will Not Occur	Will Not Occur
$\lim f(x) = +\infty$ N is large	Let $N > 0$ and set $\delta \leq \underline{\quad}$. Then if $0 < \|x - a\| < \delta$ then $f(x) > N$.	Let $N > 0$ and set $M \geq \underline{\quad}$. Then if $x > M$ then $f(x) > N$.	Let $N > 0$ and set $M \geq \underline{\quad}$. Then if $x < -M$ then $f(x) > N$.
$\lim f(x) = -\infty$ N is large	Let $N > 0$ and set $\delta \leq \underline{\quad}$. Then if $0 < \|x - a\| < \delta$ then $f(x) < -N$.	Let $N < 0$ and set $M \geq \underline{\quad}$. Then if $x > M$ then $f(x) < -N$.	Let $N > 0$ and set $M \geq \underline{\quad}$. Then if $x < -M$ then $f(x) < -N$.

	Input finite	input infinite
Output finite	δ, ϵ	M, ϵ
Output infinite	δ, N	M, N