# Math 214 Spring 2017
# Linear Algebra HW 11
# Due Friday, April 21

For all these problems, justify your answers.

1. Let $V = \mathcal{P}_n(x)$ and fix real numbers $x_0, x_1, \ldots, x_n$ be distinct real numbers. For $f, g \in V$, define
$$\langle f, g \rangle = \sum_{i=0}^{n} f(x_i) g(x_i).$$
Prove this is an inner product on $V$.
(Hint: See partial proof from class)

2. Let $w_1, \ldots, w_n$ be positive real numbers. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i.$$
Prove that this is an inner product on $\mathbb{R}^n$. (The $w_i$ are called the *weights* of the inner product).

3. Let $V = \mathcal{C}([1,3], \mathbb{R})$, with the usual inner product. Find $\|1\|$ and $\|x\|$. Find the projection of $1 + x$ onto 1 and $x$.

4. Prove the Pythagorean law: if $\mathbf{u}, \mathbf{v}$ are orthogonal, then $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

5. Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space $V$, with $\mathbf{v} \neq 0$. Let $\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.
Prove that $\langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle = 0$.

6. Let $V = \mathcal{C}([-\pi, \pi], \mathbb{R})$ with the usual inner product. Show that $\{1, \sin(x), \cos(x)\}$ is an orthogonal set. Is it orthonormal?

7. Let $V = \mathbb{R}^4$ with the dot product, and let $U = \operatorname{Span}(\{(5,3,1,0), (2,4,3,5), (1,1,1,1)\})$. Use the Gram-Schmidt process to find an orthonormal basis for $U$.

8. Let $V = \mathcal{P}_2(x)$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$.
Following the Gram-Schmidt process, convert $\{1, x, x^2\}$ into an orthonormal basis.

9. Let $V = \mathbb{R}^4$ and let $U = \operatorname{Span}(\{(3,5,2,1), (5,1,-1,-5)\}$. Find an orthonormal basis for $U^\perp$.

10. Let $\mathbf{u}_1, \mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$, and suppose $\mathbf{v}$ is a unit vector. If $\mathbf{v} \cdot \mathbf{u}_1 = 1/2$, compute $|\mathbf{v} \cdot \mathbf{u}_2|$.