Math 214 Spring 2017
Linear Algebra HW 7
Due *Wednesday*, March 22

For all these problems, justify your answers.

1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L((x, y, z)) = \begin{bmatrix} x + y + z \\ 3x - 2y + z \\ 2z \end{bmatrix}.$$

   (a) Prove that L is a linear transformation.

   (b) Find a basis for the kernel and for the image.

2. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define $E_0 : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ to be the function given by $E_0(f) = f(0)$.

   (a) Prove that E_0 is a linear transformation.

   (b) What is the kernel of E_0? What is the image?

3. Let $P_Z : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map onto the xy plane, given by $P(x, y, z) = (x, y, 0)$.

   (a) Prove that P_z is a linear transformation.

   (b) Find bases for the kernel and image of P_z.

   (c) Find a matrix for P_z with respect to the standard basis.

   (d) Prove that $P_z(P_z(\mathbf{u})) = P_z(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$. Linear transformations with this property are called *projections* and we will revisit them later. (They are also sometimes called *idempotent* if you're feeling particularly fancy).

4. (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $L((1, 2)) = (-2, 3)$ and $L((1, -1)) = (5, 2)$, what is $L((7, 5))$?

   (b) $E = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a basis for $\mathbb{R}^3$. Take the vector $\mathbf{u}$ represented by $(2, 3, 4)$ in the standard basis, and calculate $[\mathbf{u}]_E$.

   (c) $F = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ is a basis for $\mathcal{P}_3(x)$.
   Calculate $[3 + 5x - 2x^2 + x^3]_F$.

5. Let $L : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there is some real number $r \in \mathbb{R}$ such that $L(x) = rx$ for all $x \in \mathbb{R}$. (In other words, any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a scalar).

6. ($\star$) Let $L : U \to V$ be a linear transformation, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Prove that $L(U) = \text{Span}(L(E))$. That is, prove the image of L is just the span of the image of E under L.

Bonus: Don't hand in, because we won't get here by Monday, but practice for test on Friday:

($\star$) Let $U = \mathcal{P}_3(x)$, and define a linear map $D : U \to U$ by $D(f(x)) = f'(x)$. Let $E = \{1, x, x^2, x^3\}$ be a basis for U.

1. What are the kernel and image of D?

2. Find a matrix for D with respect to E and E.