Math 214 Test 1
Practice Problems

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

I will post at least some solutions soon. I probably won't get a full solutions document written up, though we'll see what happens.

Which of the following are vector spaces? Prove or disprove your answer, potentially using the subspace theorem

1. $\{(a, b, c, d) : a - b = c - d\}$
2. $\{(a, b, c, d) : a + b + c = d\}$
3. $\{(a, b, c) : a^2 = bc\}$
4. $\{(a, b, c, d) : 5a - 3b = 2c - 2d\}$
5. $\{a_0 + a_1x + a_2x^2 + a_3x^3 : a_2 = 2\}$
6. $\{f(x) : f(0) = 5\}$
7. $\{f(x) : f(5) = 0\}$

Write u as a linear combination of vectors in S, or prove you cannot

1. $\mathbf{u} = (5, 2, 1)$, $S = \{(1, 2, 3), (3, 1, 1)\}$
2. $\mathbf{u} = (2, 3, 2)$, $S = \{(1, 2, 3), (3, 4, 1)\}$
3. $\mathbf{u} = x^3 - x + 1$, $S = \{1 + x, 3 + x^2, 3x^2 + x^3\}$
4. $\mathbf{u} = x^3 + 4x^2 + 2x + 5$, $S = \{1 + x, 3 + x^2, 3x^2 + x^3\}$

For each of the following sets, check:

- Does it span the (implicitly given) vector space?
- Is it linearly independent?
- Is it a basis?

1. $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$
2. $S = \{(2, 1, -2), (3, 2, -2), (2, 2, 0)\}$
3. $S = \{(1, 0, 0, 1), (0, 1, 0, 0), (2, 3, 0, 2)\}$
4. $S = \{(1, 5, 2), (3, 1, 4), (-1, 3, 7), (2, 8, 1)\}$
5. $S = \{1 + x^2, 1 + x^3, x - x^2, 5 + x^2 - 4x^3\}$
6. $S = \{1 + 2x, x + 2x^2, x^2 + 2x^3, 2 + x^3\}$

Bases

1. Find a basis for $\mathbb{R}^3$ containing $(-1,3,2)$ and $(5,4,1)$.

2. Find a basis for $\mathbb{R}^3$ containing $(7,1,-3)$ and $(1,1,1)$.

   It looks like $(1,0,0)$ is not in the span of $\{(7,1,-3),(1,1,1)\}$, so we test:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}7\\1\\-3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}7a+b\\a+b\\b-3a\end{bmatrix}$$

   giving

$$7a+b=1 \qquad a+b=0 \qquad b-3a=0.$$

   This gives us $a=-b$ so $b+3b=0$ implies $b=0$ and thus $a=0$, so we have $0=1$, a contradiction. Thus $(1,0,0)$ is not in the span, so by basis padding $\{(7,1,-3),(1,1,1),(1,0,0)\}$ is a basis for $\mathbb{R}^3$.

3. Find a basis for $\mathbb{R}^4$ containing $(1,2,3,4),(1,1,1,1)$, and $(0,0,1,1)$.

4. Find a basis for $\mathcal{P}_3(x)$ containing $1+3x^3, x^2-x, 6-2x$.

5. Find a basis for $\mathbb{R}^3$ that is a subset of $\{(1,1,1),(2,4,6),(7,-1,2),(2,5,-2),(3,-6,4)\}$.

6. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(1,3),(2,4),(1,1)\}$.

7. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(-1,4),(7,-2),(3,6)\}$.

8. Find a basis for $\mathcal{P}_2(x)$ that is a subset of $\{1+x, 3+x^2, 4+3x+2x^2, x^2-7x\}$.

Proofs

1. Suppose U,W are subspaces of some vector space V. Prove that the set $U+W=\{\mathbf{u}+\mathbf{w} : \mathbf{u}\in U, \mathbf{w}\in W\}$ is a subspace of V.

   Bonus: what is the space $U+U$?

2. Suppose $S=\{\mathbf{v}_1,\dots,\mathbf{v}_n\}\subseteq V$ is linearly independent. Show that $T=\{\mathbf{v}_1,\mathbf{v}_2-\mathbf{v}_1,\dots,\mathbf{v}_n-\mathbf{v}_{n-1}\}$ is linearly independent.

Bonus to stretch your brain

1. Find a subset $U\subset\mathbb{R}^2$ that is closed under scalar multiplication but is not a subspace.

2. Find a subset $U\subset\mathbb{R}^2$ that is closed under addition but is not a subspace.

3. Can you find a basis for $\mathcal{P}_3(x)$ such that no element of the basis has degree 3?

4. Can you find a basis for $\mathcal{P}_3(x)$ such that no element of the basis has degree 2?