3 Limit Laws

3.1 Algebra of Limits

We've spent the first couple weeks computing and calculating limits by hand, straight from the definition. By now this has gotten fairly tedious. We'd like to develop some rules, tools, and principles that will allow us to compute limits of functions without going back to the definition.

Our approach is to break this into two steps. First, we'll compute a few very simple types of limits. Then we'll prove rules that will let us decompose more complicated limits into combinations of these simples limts.

First our basic building blocks:

Lemma 3.1 (Identity). *Let a be a real number. Then $\lim_{x\to a} x = a$.*

Proof. Let $\epsilon > 0$ and let $\delta = \epsilon$. If $|x - a| < \delta$, then $|x - a| < \delta = \epsilon$. □

Lemma 3.2 (Almost Identical Functions). *If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ whenever one limit exists.*

Proof. Suppose $\lim_{x\to a} f(x) = L$. Let $\epsilon > 0$; then there is some δ_1 such that if $0 < |x-a| < \delta_1$ then $|f(x) - L| < \epsilon$. Then let $\delta < d, \delta_1$. If $0 < |x - a| < \delta$ then $g(x) = f(x)$, and thus

$$|g(x) - L| = |f(x) - L| < \epsilon.$$

□

Exercise 3.3 (Constants). *Prove that if a, c are real numbers, then $\lim_{x\to a} c = c$.*

Proof. See Homework 4. □

Now we can progress to some more complex results that allow us to patch our simple results together. I will prove a couple of these, but most of the proofs are more tedious than enlightening in our context.

Proposition 3.4. *Suppose c is a constant real number, and f and g are functions such that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ exist. Then*

1. *(Additivity)* $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.

Proof. Let $\epsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - L_1| < \epsilon/2$, and if $0 < |x-a| < \delta_2$ then $|g(x) - L_2| < \epsilon/2$.

Let $\delta \leq \delta_1, \delta_2$. Then if $0 < |x-a| < \delta$, we compute

$$|f(x)+g(x)-(L_1+L_2)| = |(f(x)-L_1)+(g(x)-L_2)| \leq |f(x)-L_1|+|g(x)-L_2| < \epsilon/2+\epsilon/2 = \epsilon.$$

□

2. *(Scalar multiples)* $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$

Proof. If $c = 0$ then the left hand side is $\lim_{x\to a} 0 = 0$ and the right hand side is $0L_1 = 0$ so the equality holds.

If $c \neq 0$, then let $\epsilon > 0$. Then by definition of limit, there exists some δ so that if $0 < |x-a| < \delta$ then $|f(x) - L_1| < \epsilon/c$.

Then if $0 < |x-a| < \delta$, we have

$$|cf(x) - cL_1| = c|f(x) - L_1| < c(\epsilon/c) = \epsilon,$$

which is what we wanted to show. □

3. *(Products)* $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

Proof. Let $\epsilon > 0$. Then there exist δ_1, δ_2 such that

- if $0 < |x-a| < \delta_1$ then $|f(x) - L_1| < \epsilon/(2|L_2|), 1$,
- and if $0 < |x-a| < \delta_2$ then $|g(x) - L_2| < \epsilon/(2|L_1| + 2)$.

Set $\delta \leq \delta_1, \delta_2$. Then if $0 < |x-a| < \delta$, we compute

$$\begin{aligned}
|f(x)g(x) - L_1L_2| &= |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1L_2| \\
&\leq |f(x)g(x) - f(x)L_2| + |f(x)L_2 - L_1L_2| \\
&= |f(x)| \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
&= |f(x) - L_1 + L_1| \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
&\leq (|f(x) - L_1| + |L_1|) \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
&< (1 + |L_1|)\left(\epsilon/(2|L_1| + 2)\right) + |L_2| \cdot \epsilon/(2Ł_2|) \\
&= \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

□

4. *(Quotients) That last rule also works with division if that makes sense: if* $\lim_{x \to a} g(x) \neq 0$*, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

Proof. I'm not going to prove this because it's really long and annoying and not very informative. It's a lot like the last proof except more tedious. If you're feeling masochistic you can probably prove it yourself. □

5. *(Exponents) The rule for multiplication extends to exponentials:* $\lim_{x \to a}(f(x)^n) = (\lim_{x \to a} f(x))^n$*. Also roots:* $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$*, assuming all the functions make sense.*

Proof. We're only going to prove this for the case of $f(x)^n$ where n is a positive integer. The other proofs are basically the same, but this has less bookkeeping.

$$\begin{aligned}
\lim_{x \to a} f(x)^n &= \lim_{x \to a} f(x) \cdot f(x)^{n-1} \\
&= \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} f(x)^{n-1}\right) && \text{by the rule on products} \\
&\vdots \\
&= \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} f(x)\right) \cdot \dots \cdot \left(\lim_{x \to a} f(x)\right) \\
&= \left(\lim_{x \to a} f(x)\right)^n
\end{aligned}$$

Formally we should write this up as a proof by induction. □

Example 3.5. 1.

$$\begin{aligned}
\lim_{x \to 1} x^3 &= \left(\lim_{x \to 1} x\right)^3 && \text{Exponents} \\
&= 1^3 && \text{Identity} \\
&= 1
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to 1}(x+1)^3 - 2 &= \lim_{x\to 1}(x+1)^3 - \lim_{x\to 1} 2 && \text{Additivity} \\
&= \left(\lim_{x\to 1}(x+1)\right)^3 - 2 && \text{Exponents and Constants} \\
&= \left(\lim_{x\to 1} x + \lim_{x\to 1} 1\right)^3 - 2 && \text{Additivity} \\
&= (1+1)^3 - 2 && \text{Identity and Constants} \\
&= 2^3 - 2 = 8 - 2 = 6.
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2}{x} &= \frac{\lim_{x\to 1} x^2}{\lim_{x\to 1} x} && \text{Quotients} \\
&= \frac{(\lim_{x\to 1} x)^2}{\lim_{x\to 1} x} && \text{Exponents} \\
&= \frac{1^2}{1} && \text{Identity} \\
&= 1/1 = 1.
\end{aligned}$$

We can also approach this problem a different way, since this function is just the same as x everywhere except at 0:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2}{x} &= \lim_{x\to 1} x && \text{Almost Identical Functions} \\
&= 1 && \text{Identity}
\end{aligned}$$

4.

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2}{x} &= \lim_{x\to 0} x && \text{Almost Identical Functions} \\
&= 0
\end{aligned}$$

Unlike the previous problem, we *cannot* use the Quotient property here because the bottom approaches zero. Compare:

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{x^2} &= \lim_{x\to 0} \frac{1}{x} && \text{Almost Identical Functions} \\
&\neq \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} x}
\end{aligned}$$

The last step doesn't work because now we're dividing by zero, which we can never do. This limit is in fact $\pm\infty$, and we'll look at how to show that without a proof from the definition soon.

Of course, even showing all these steps gets tedious, so you don't have to do that unless I explicitly ask you to. It's useful to be able to when you want to check your work carefully, or when you're working with something particularly tricky.

Poll Question 3.1.1. What is $\lim_{x\to 4} \frac{x^2-8x+16}{x-4}$?

$$\begin{aligned}
\lim_{x\to 4} \frac{x^2-8x+16}{x-4} &= \lim_{x\to 4} \frac{(x-4)(x-4)}{x-4} && \text{arithmetic} \\
&= \lim_{x\to 4} x-4 && \text{Almost Identical Functions} \\
&= \lim_{x\to 4} x - \lim_{x\to 4} 4 && \text{Additivity} \\
&= 4-4 && \text{Identity and Constants} \\
&= 0.
\end{aligned}$$

1. $\lim_{x\to -2} \frac{(x+1)^2-1}{x+2} = \lim_{x\to -2} \frac{x^2+2x+1-1}{x+1} = \lim_{x\to -2} \frac{x(x+2)}{x+2} = \lim_{x\to -2} x = -2.$

   Note that $\frac{x(x+2)}{x+2} \neq x$, but their limits a 0 are the same because the functions are the same near 0 (and in fact everywhere except at 0).

2. What is $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$?

   We use a trick called multiplication by the conjugate, which takes advantage of the fact that $(a+b)(a-b) = a^2-b^2$. This trick is used *very often* so you should get comfortable with it.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} &= \lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3} \\
&= \lim_{x\to 0} \frac{(9+x)-3}{x(\sqrt{9+x}+3)} = \lim_{x\to 0} \frac{x}{x(\sqrt{9+x}+3)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{\lim_{x\to 0} \sqrt{9+x}+3} = \frac{1}{6}.
\end{aligned}$$

3. What is $\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2}$?

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\
&= \lim_{x\to 1} -(\sqrt{5-x}+2) = -4.
\end{aligned}$$

Proposition 3.6 (Two-sided limits). $\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$. *In particular the two-sided limit exists if and only if both one-sided limits exist and are the same.*

Proof. Suppose $\lim_{x\to a} f(x) = L$. Let $\epsilon > 0$; then there is a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x)-L| < \epsilon$. Then if $a-\delta < x < a$, we know that $|x-a| < \delta$ and thus $|f(x)-L| < \epsilon$, so $\lim_{x\to a^-} f(x) = L$ by definition. Similarly if $a < x < a+\delta$ then $0 < |x-a| < \delta$ and thus $|f(x)-L| < \epsilon$, so $\lim_{x\to a^+} f(x) = L$.

Conversely, suppose $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$. Let $\epsilon > 0$. Then there is a δ_1 so that if $a-\delta_1 < x < a$ then $|f(x)-L| < \epsilon$. And there is a δ_2 so that if $a < x < a+\delta_2$ then $|f(x)-L| < \epsilon$.

So let $\delta \le \delta_1, \delta_2$. Then if $0 < |x-a| < \delta$, either $a-\delta_1 < x < a$ or $a < x < a+\delta_2$, and either way $|f(x)-L| < \epsilon$. ☐

Example 3.7. Let $f(x) = \begin{cases} \frac{x^2+4x+3}{x+1} & x < -1 \\ x^2+1 & x > -1 \end{cases}$. What is $\lim_{x\to 2} f(x)$?

We can take the left and right limits separately.

$$\begin{aligned}
\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{x^2+4x+3}{x+1} = \lim_{x\to -1^-} x+3 = (-1)+3 = 2. \\
\lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} x^2+1 = 2.
\end{aligned}$$

Since the one-sided limits exist and are equal, we have $\lim_{x\to -1} f(x) = 2$.

Example 3.8. Let $g(x) = \begin{cases} \frac{x^2+x-6}{x-2} & x < 2 \\ \frac{\sqrt{3x+10}-4}{x-2} & x > 2 \end{cases}$. What is $\lim_{x\to 2} g(x)$?

We can take the left and right limits separately. We have

$$\begin{aligned}
\lim_{x\to 2^-} g(x) &= \lim_{x\to 2^-} \frac{x^2+x-6}{x-2} = \lim_{x\to 2^-} x+3 = 5 \\
\lim_{x\to 2^+} g(x) &= \lim_{x\to 2^+} \frac{\sqrt{3x+10}-4}{x-2} = \lim_{x\to 2^+} \frac{\sqrt{3x+10}-4}{x-2}\frac{\sqrt{3x+10}+4}{\sqrt{3x+10}+4} \\
&= \lim_{x\to 2^+} \frac{3x+10-16}{(x-2)(\sqrt{3x+10}+4)} = \lim_{x\to 2^+} \frac{3(x-2)}{(x-2)(\sqrt{3x+10}+4)} \\
&= \lim_{x\to 2^+} \frac{3}{\sqrt{3x+10}+4} = \frac{3}{8}.
\end{aligned}$$

The left and right limits are not equal, so the limit does not exist.

3.2 Infinite limits

Now we want to talk about infinite limits, that is, limits where the output increases without bound.

Lemma 3.9. *Let $f(x), g(x)$ be defined near a, such that $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \pm\infty.$$

Further, assuming $c > 0$ then the limit is $+\infty$ if and only if $g(x) \geq 0$ near a, and the limit is $-\infty$ if and only if $g(x) \leq 0$ near a. If $c < 0$ then the opposite is true.

Remark 3.10. If the limit of the numerator is zero, then this lemma is *not useful.* That is one of the "indeterminate forms" which requires more analysis before we can compute the limit completely.

Proof. Let $N > 0$. Then there's a δ_1 so that if $0 < |x-a| < \delta_1$ then $|f(x)-c| < c/2$ and thus in particular $f(x) > c/2$. And there's a δ_2 so that if $0 < |x-a| < \delta_2$ then $|g(x)| < c/2N$.

Let $\delta \leq \delta_1, \delta_2$. Then if $0 < |x-a| < \delta$ we have

$$\left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} > \frac{c/2}{|g(x)|} > \frac{c/2}{c/2N} = N.$$

□

Example 3.11. What is $\lim_{x\to 3} \frac{-1}{\sqrt{x-3}}$? We see the top goes to 1 and the bottom goes to 0, so the limit is $\pm\infty$. Since the denominator is always positive and the numerator is negative, the limit is $-\infty$.

We have to be careful while working these problems: the limit laws that work for finite limits don't always work here, since the limit laws assume that the limits exist, and these do not. In particular, adding and subtracting infinity *does not work*. Instead, we need to arrange the function into a form where we can use lemma 3.9.

Example 3.12. We already know that $\lim_{x \to 0} 1/x = \pm\infty$.

1. If we take $\lim_{x \to 0} 1/x - 1/x$, we could say the limit is $\pm\infty - \pm\infty$, but this is silly—the limit is actually 0.

2. In contrast, $\lim_{x \to 0} 1/x + 1/x = \lim_{x \to 0} 2/x = \pm\infty$. We don't add the infinities together.

3. And $\lim_{x \to 0} 1/x + 1/x^2$ is the trickiest. We have a $\pm\infty$ plus a $+\infty$. But again we can't add infinities—we need to combine them into one term.
$$\lim_{x \to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x \to 0} \frac{x+1}{x^2} = +\infty$$
since the numerator approaches 1 and the denominator approaches 0, but is always positive.

   We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

We organize our thinking about these situations in terms of the "indeterminate forms", which are: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty \pm \infty, 1^\infty, \infty^0$. Notice that none of these are actual numbers, and they can never be the correct answer to pretty much any question.

More importantly, indeterminate forms don't even tell us what the answer should be; if plugging in gives you one of those forms, the true limit could potentially be pretty much anything. We have to do more work to get our functional expression into a determinate form. As a general rule, we use algebraic manipulations to get a form of $\frac{0}{0}$, then factor out and cancel $(x - a)$ until either the numerator or the denominator is no longer 0.

Remark 3.13. Neither $\frac{0}{1}$ nor $\frac{1}{0}$ is an indeterminate form. $\frac{0}{1}$ is just a number, equal to 0. $\frac{1}{0}$ is not a number and is never the correct answer to a question, but it's also not indeterminate. By lemma 3.9, if $\lim f(x) = 1$ and $\lim g(x) = 0$ then $\lim f(x)/g(x) = \pm\infty$.

Similarly, $\frac{0}{\infty}$ and $\frac{\infty}{0}$ are also not numbers but not indeterminate. The first suggests the limit is 0; the second suggests the limit is $\pm\infty$.

The form $\infty \cdot \infty$ mostly works fine, and gives you another ∞ whose sign depends on the signs of the ∞s you're multiplying. But again, $\infty \cdot \infty$ is never the actual answer to any actual question.

Example 3.14. What is $\lim_{x \to -2} \frac{1}{x+2} + \frac{2}{x(x+2)}$? This looks like $\infty + \infty$ so we have to be careful. We have

$$\begin{aligned}
\lim_{x \to -2} \frac{1}{x+2} + \frac{2}{x(x+2)} &= \lim_{x \to -2} \frac{x}{x+2} + \frac{2}{x(x+2)} \\
&= \lim_{x \to -2} \frac{x+2}{x(x+2)} = \lim_{x \to -2} \frac{1}{x} = \frac{-1}{2}.
\end{aligned}$$

Example 3.15. $\lim_{x \to 3^+} \frac{1}{(x-3)^3} = +\infty$: the limit of the top is 1, and the limit of the bottom is 0, so the limit is $\pm\infty$. But when $x > 3$ the denominator is ≥ 0, so the limit is in fact $+\infty$. Conversely $\lim_{x \to 3^-} \frac{1}{(x-3)^3} = -\infty$ since when $x < 3$ we have $(x-3)^3 \leq 0$.

$\lim_{x \to -1^+} \frac{1}{(x+1)^4} = +\infty$. And $\lim_{x \to -1^-} \frac{1}{(x+1)^4} = +\infty$. Thus $\lim_{x \to -1} \frac{1}{(x+1)^4} = +\infty$.

3.3 Limits at Infinity

Something we haven't discussed so far is the idea of limits "at" infinity, which answers the question "what happens to $f(x)$ when x gets very big?" We can formally define this in terms of ϵ.

Definition 3.16. Let f be a function defined for (a, ∞) for some number a. We write

$$\lim_{x \to +\infty} f(x) = L$$

to indicate that when x is large enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x) - L| < \epsilon$.

We can write similar definitions for $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \pm\infty} f(x)$, and talk about when these limits are themselves $\pm\infty$. But here we'll skip over the formal definition and simply think informally.

For the most part, we can use all of our limit laws from sections 3.1 and 3.2, but we need to add one additional principle. We can't really use the principle of identity, since we can't do arithmetic with infinity. Instead, we use the following fact:

Fact 3.17. $\lim_{x \to \pm\infty} \frac{1}{x} = 0$.

This combined with the limit laws we already have is enough to do pretty much any calculuation here.

Example 3.18. If we want to calculuate $\lim_{x \to +\infty} \frac{1}{\sqrt{x}}$, we see that

$$\lim_{x \to +\infty} \frac{1}{\sqrt{x}} = \sqrt{\lim_{x \to +\infty} \frac{1}{x}} = \sqrt{0} = 0.$$

Example 3.19. What is $\lim_{x\to+\infty}\frac{x}{x^2+1}$?

This problem illustrates the primary technique we'll use to solve infinite limits problems. It's difficult to deal with problems that have variables in the numerator and denominator, so we want to get rid of at least one. Thus we will divide out by xs on the top and the bottom until one has none left:

$$\lim_{x\to+\infty}\frac{x}{x^2+1}=\lim_{x\to+\infty}\frac{x/x}{x^2/x+1/x}=\lim_{x\to+\infty}\frac{1}{x+\frac{1}{x}}=\lim_{x\to+\infty}\frac{1}{x}=0.$$

Example 3.20. Some more examples of this technique:

$$\lim_{x\to-\infty}\frac{x}{x+1}=\lim_{x\to-\infty}\frac{1}{1+\frac{1}{x}}=\lim_{x\to-\infty}\frac{1}{1}=1.$$

$$\lim_{x\to-\infty}\frac{x}{3x+1}=\lim_{x\to-\infty}\frac{1}{3+\frac{1}{x}}=\frac{1}{3}.$$

Example 3.21. What is $\lim_{x\to+\infty}\frac{x^{3/2}}{\sqrt{9x^3+1}}$? This one is a bit tricky. We want to divide the top and bottom by $x^{3/2}$. Then we can pull the factor *inside* the square root sign.

$$\lim_{x\to+\infty}\frac{x^{3/2}}{\sqrt{9x^3+1}}=\lim_{x\to+\infty}\frac{1}{\sqrt{9+1/x^{3/2}}}=\frac{1}{\sqrt{9+0}}=\frac{1}{3}.$$

Example 3.22. Sometimes it's a bit harder to see how this works. For instance, what is $\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+1}}$? It's not obvious, but we use the same technique:

$$\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to+\infty}\frac{x/x}{\sqrt{x^2+1}/x}=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2/x^2+1/x^2}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}}=1.$$

Example 3.23. What is $\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}$?

We can do the same thing, but we have to be *very careful.* Remember that if $x<0$ then $\sqrt{x^2}\neq x$! Instead, $x=-\sqrt{x^2}$. Thus we have

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to-\infty}\frac{1}{\sqrt{x^2+1}/x}=\lim_{x\to-\infty}\frac{1}{\sqrt{x^2+1}/(-\sqrt{x^2})}=\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x^2}}}=-1.$$

When we encounter new functions, one of the ways we will often want to characterize them is by computing their limits at $\pm\infty$. Sometimes these limits do not exist.

Example 3.24. $\lim_{x\to+\infty}\sin(x)$ does not exist, since the function oscillates rather than settling down to one limit value.

$\lim_{x\to+\infty}x\sin(x)$ also does not exist; this function oscillates more and more wildly as x increases.

We shall see later on that $\lim_{x\to+\infty}\frac{1}{x}\sin(x)$ exists.

Another technique that will also often appear in these limits is combining a sum or difference into one fraction. If we have a sum of two terms that both have infinite limits, we need to combine or factor them into one term to see what is happening.

Example 3.25. What is $\lim_{x\to-\infty} x - x^3$?

Each term goes to $-\infty$, so this is a difference of infinities and thus indeterminate. But we can factor: $\lim_{x\to-\infty} x(1-x^2)$. The first term goes to $-\infty$ and the second term also goes to $-\infty$, so we expect that their product will go to $+\infty$. Thus $\lim_{x\to-\infty} x - x^3 = +\infty$.

To be precise, I should compute:

$$\lim_{x\to-\infty} x - x^3 = \lim_{x\to-\infty} \frac{x-x^3}{1} = \lim_{x\to-\infty} \frac{1/x^2 - 1}{1/x^3}.$$

We see the limit of the top is -1 and the limit of the bottom is 0, so the limit of the whole is $\pm\infty$. In fact the bottom will always be negative (since $x \to -\infty$), and thus the limit is $+\infty$.

Example 3.26. What is $\lim_{x\to+\infty} \sqrt{x^2+1} - x$?

We might want to try to use limit laws here, but we would get $+\infty - +\infty$ which is not defined (and is one of the classic indeterminate forms). Instead we need to combine our expressions into one big fraction.

$$\begin{aligned}\lim_{x\to+\infty} \sqrt{x^2+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to+\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}+x}.\end{aligned}$$

The numerator is 1 and the denominator approaches $+\infty$ so the limit is 0. This tells us that as x increases, x and $\sqrt{x^2+1}$ get as close together as we wish.

You may have noticed the appearance of our old friend, multiplication by the conjugate. We will often use that technique in this sort of problem.

Example 3.27. What is $\lim_{x\to+\infty} \sqrt{x^2+x+1} - x$?

$$\begin{aligned}\lim_{x\to+\infty} \sqrt{x^2+x+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+x+1} - x\right) \frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} \\ &= \lim_{x\to+\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+x+1}+x} \\ &= \lim_{x\to+\infty} \frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1} = \frac{1}{2}.\end{aligned}$$

3.4 Trigonometric Limits and the Squeeze Theorem

We now want to look at limits of trigonometric functions. We will state the following proposition without proof:

Proposition 3.28. *If a is a real number, then* $\lim_{x\to a} \sin(x) = \sin(a)$ *and* $\lim_{x\to a} \cos(x) = \cos(a)$.

In fact, since trigonometric functions are just ways of combining sine and cosine, essentially all trigonometric functions behave this way where defined.

Recall we previously saw that $\lim_{x\to 0} \sin(1/x)$ does not exist.

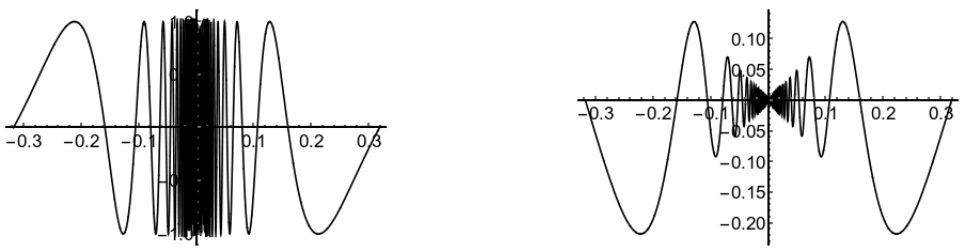


Figure 3.1: Left: graph of $\sin(1/x)$, Right: graph of $x\sin(1/x)$

In contrast, from the graph it appears that $\lim_{x\to 0} x\sin(1/x)$ does exist. To prove this we will need:

Theorem 3.29 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, *then* $\lim_{x\to a} g(x) = L$.

Now we see that since $-1 \leq \sin(1/x) \leq 1$, we can write

$$-|x| \leq x\sin(1/x) \leq |x|$$

But $\lim_{x\to 0}(-|x|) = \lim_{x\to 0} |x| = 0$ and so by the squeeze theorem, $\lim_{x\to 0} x\sin(1/x) = 0$.

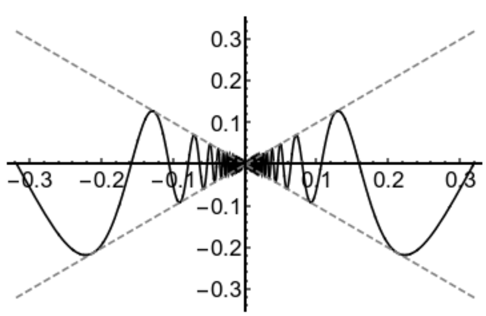


Figure 3.2: A graph of $x\sin(1/x)$ with $|x|$ and $-|x|$

Example 3.30. Show that $\lim_{x\to 3}(x-3)\frac{x^2}{x^2+1} = 0$.

We know that $0 \leq x^2 \leq x^2+1$ and so $0 \leq \frac{x^2}{x^2+1} \leq 1$ for any x. Thus

$$-|x-3| \leq (x-3)\frac{x^2}{x^2+1} \leq |x-3|.$$

(We need the absolute value signs so that this statement is true if $x < 3$ as well).

Then $\lim_{x\to 3}|x-3| = \lim_{x\to 3} -|x-3| = 0$, and so by the squeeze theorem $\lim_{x\to 3}(x-3)\frac{x^2}{x^2+1} = 0$.

Example 3.31. What is

$$\lim_{x\to 1} \frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}?$$

The top goes to zero and the bottom is bounded, so this looks like a squeeze theorem problem.

We know that $-1 \leq \sin\left(\frac{1}{x-1}\right) \leq 1$ and so $1 \leq 2+\sin\left(\frac{1}{x-1}\right) \leq 3$, and thus

$$\begin{aligned} 1 &\geq \frac{1}{2+\sin\left(\frac{1}{x-1}\right)} \geq \frac{1}{3} \\ |x-1| &\geq \frac{|x-1|}{2+\sin\left(\frac{1}{x-1}\right)} \geq \frac{|x-1|}{3} \\ |x-1| &\geq \left|\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}\right| \geq \frac{|x-1|}{3} \end{aligned}$$

since the denominator is always positive. But $\lim_{x\to 1}|x-1| = \lim_{x\to 1}\frac{|x-1|}{3} = 0$, so by the squeeze theorem

$$\lim_{x\to 1}\left|\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}\right| = 0.$$

But if the absolute value goes to zero, the function goes to zero, and thus

$$\lim_{x\to 1}\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)} = 0.$$

Figuring out the bounds here can be kind of tricky, but we can simplify things with one additional tool:

Proposition 3.32. *Suppose f is defined near a. Then $\lim_{x\to a} f(x) = 0$ if and only if $\lim_{x\to a}|f(x)| = 0$.*

This helps us because it lets us work entirely inside the absolute value signs, thus letting us avoid any concerns about the sign.

Example 3.33 (Continued)**.** Above we could reason that

$$\frac{1}{3} \leq \left| \frac{1}{2 + \sin\left(\frac{1}{x-1}\right)} \right| \leq 1$$
$$\frac{|x-1|}{3} \leq \left| \frac{x-1}{2 + \sin\left(\frac{1}{x-1}\right)} \right| \leq |x-1|.$$

Then $\lim_{x \to 1} \frac{|x-1|}{3} = \lim_{x \to 1} |x-1| = 0$, so by the squeeze theorem,

$$\lim_{x \to 1} \left| \frac{x-1}{2 + \sin\left(\frac{1}{x-1}\right)} \right| = 0.$$

Thus we know that

$$\lim_{x \to 1} \frac{x-1}{2 + \sin\left(\frac{1}{x-1}\right)} = 0.$$

Example 3.34. What is $\lim_{x \to -1} (x+1) \cos\left(\frac{x^5 - 3x^2 + e^x - 1700 + (2+x)^{(1+x)^x}}{(x+1)^{27.2}} \right)$?

This looks complicated but is actually quite simple. $-1 \leq \cos(y) \leq 1$ for any y, including $y = x^5 - 3x^2 + e^x - 1700 + x^{x^x}$. Thus we have

$$0 \leq |\cos(y)| \leq 1$$
$$0 \leq |(x+1)\cos(y)| \leq |x+1|.$$

Then we know that $\lim_{x \to -1} 0 = \lim_{x \to -1} |x+1| = 0$. Thus by the squeeze theorem,

$$\lim_{x \to -1} |(x+1)\cos(x^5 - 3x^2 + e^x - 1700 + x^{x^x})| = 0,$$

and thus

$$\lim_{x \to -1} (x+1)\cos(x^5 - 3x^2 + e^x - 1700 + x^{x^x}) = 0.$$

Example 3.35. What is

$$\lim_{x \to 0} \frac{x-1}{2 + \sin\left(\frac{1}{x-1}\right)}?$$

This is a trick question. Here we have no concerns about zeroes in the denominator or points outside of the domain, we can repeatedly apply limit laws:

$$\lim_{x \to 0} \frac{x-1}{2 + \sin\left(\frac{1}{x-1}\right)} = \frac{\lim_{x \to 0}(x-1)}{\lim_{x \to 0} 2 + \sin\left(\frac{1}{x-1}\right)}$$
$$= \frac{-1}{2 + \sin\left(\lim_{x \to 0} \frac{1}{x-1}\right)}$$
$$= \frac{-1}{2 + \sin(-1)} = \frac{-1}{2 - \sin(1)}.$$

Remark 3.36. Notice that we don't conclude that since $f(x) \le g(x) \le h(x)$ then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)$. This is in fact not always true; it's only true if the middle limit exists, which is what we're trying to prove! So we just compute the outer two limits, and then invoke the squeeze theorem.

Example 3.37. $\lim_{x\to+\infty} \frac{\sin(x)}{x}$ exists, by the squeeze theorem.

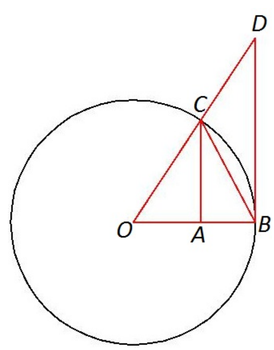
For large x we have $\frac{-1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$, and $\lim_{x\to+\infty} \frac{-1}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$. So by the squeeze theorem $\lim_{x\to+\infty} \frac{\sin(x)}{x} = 0$.

You might notice this is *exactly the same proof* we gave for $\lim_{x\to 0} x \sin(1/x)$. This is not a coincidence, since the two functions are the same after the substitution $y = 1/x$.

There is one more important limit involving sin:

Proposition 3.38 (Small Angle Approximation)**.**

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$



Proof. We'll assume θ is small and positive; this all still works if θ is small and negative, with different signs. Our diagram is of a circle with radius 1.

Let θ be the measure of angle AOC in our diagram. Observe that $\sin\theta$ is precisely the length of the line segment AC by definition, and so triangle BOC has area $\sin\theta/2$. The area of the entire circle is π and so the area of the wedge from B to C is $\pi\theta/2\pi = \theta/2$. Since the triangle is contained in the wedge, we have $\sin\theta/2 \le \theta/2$ and thus $\sin\theta/\theta \le 1$.

Note that AC is $\sin\theta$ and AO is $\cos\theta$, so AC over AO is $\sin(\theta)/\cos(\theta) = \tan(\theta)$. By similarity, we have DB $= \tan\theta$, and the area of triangle BOD is $\tan\theta/2$. Since the wedge from B to C is contained in this triangle, we have $\theta/2 \le \tan\theta/2$ and thus $\cos\theta \le \sin\theta/\theta$.

Thus $\cos\theta \le \frac{\sin\theta}{\theta} \le 1$. But $\lim_{\theta\to 0} \cos\theta = 1$, so by the squeeze theorem we have

$$1 \le \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \le 1$$

and thus get the desired result. □

Example 3.39. $\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$.

Example 3.40. What is $\lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x}$?

We can write

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x} &= \lim_{x\to 0} \frac{\sin(4x)/4x \cdot \sin(6x)/6x \cdot 24x^2}{\sin(2x)/2x \cdot 2x \cdot x} \\
&= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{\sin 6x}{6x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{24x^2}{2x^2} \\
&= 1 \cdot 1 \cdot 1 \cdot 12 = 12.
\end{aligned}$$

Here we are simply pairing off the $\sin(y)$'s with ys and then collecting the remainder into the last term.

Example 3.41. $\lim_{x\to 0} \csc x = \pm\infty$ since $\csc(x) = \frac{1}{\sin(x)}$; the top goes to 1, and the bottom goes to 0 two-sidedly.

$\lim_{x\to 1}(x-1)\csc(x-1) = \lim_{x\to 1} \frac{x-1}{\sin(x-1)} = 1$ by the small angle approximation.

Poll Question 3.4.1. What is $\lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)}$?

When we see a tangent in a problem, it is often helpful to rewrite it in terms of sin and cos. We can then collect terms:

$$\begin{aligned}
\lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)} &= \lim_{x\to 0} \frac{x\sin(2x)}{\sin(3x)/\cos(3x)} \\
&= \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \frac{\sin(2x)\cos(3x)}{3} = 1 \cdot \frac{0}{3} = 0.
\end{aligned}$$

Example 3.42. What is $\lim_{x\to 3} \frac{\sin(x-3)}{x-3}$?

This is a small angle approximation again, since $x - 3$ is approaching zero. Thus the limit is 1.

Example 3.43. What is $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$?

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}
\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} &= \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9} \\
&= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x + 3 = 1 \cdot (3+3) = 6.
\end{aligned}$$

Example 3.44. What is $\lim_{x\to 0} \frac{1-\cos x}{x}$?

We can see that the limits of the top and the bottom are both 0, so this is an indeterminate form. We can't use the small angle approximation directly because there is no sin here at all. But we can fix that by multiplying by the conjugate.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0}\frac{\sin^2(x)}{x(1+\cos(x))} \\ &= \lim_{x\to 0}\frac{\sin(x)}{1+\cos(x)} = \frac{0}{2} = 0.\end{aligned}$$