7 Implicit Functions

7.1 Implicit Derivatives

We defined a function as a rule, that takes some input and gives some output. Usually we give you the rule explicitly, as when we say $y = x^2 - 1$. But sometimes you only know facts about the rule, such as $y^2 + x^2 = 1$ (which describes the unit circle). Sometimes these facts will describe one function uniquely, and sometimes they won't. (This comes up a lot in solving actual problems in physics and economics and other fields).

Regardless of where we get an equation like this, we know that both sides are equal, so the derivatives of both sides are equal. So using the chain rule and thinking of y as a function of x, we can simply take derivatives of both sides, and then do some algebra to find y'.

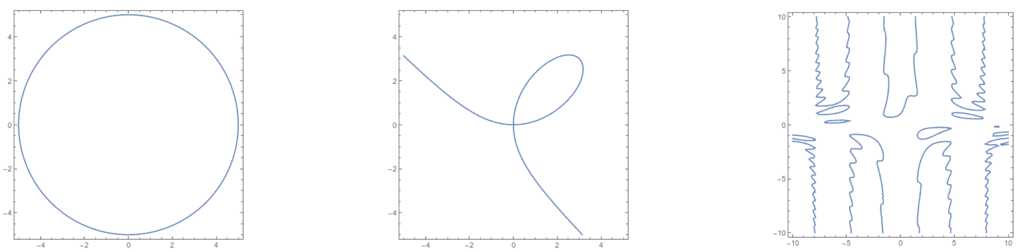

Figure 7.1: Left: The circle $x^2 + y^2 = 25$. Center: the folium of Descartes $x^3 + y^3 = 6xy$. Right: $y\cos(x) = 1 + \sin(xy)$

If we want to find tangent lines for these curves, we can use implicit differentiation. Essentially, we take the derivative of both sides of the equation, treating y as a function of x and applying the chain rule.

Example 7.1.

- If $x^2 + y^2 = 25$, then

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(25) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-x}{y}. \end{aligned}$$

  Thus at the point, say, $(3, 4)$ (check that this is on the circle!), we have that $\frac{dy}{dx}(3, 4) = \frac{-3}{4} = -3/4$. Thus the equation of the line tangent to the circle at $(3, 4)$ is $y - 4 = -\frac{3}{4}(x - 3)$.

- If $x^3 + y^3 = 6xy$, then

$$\begin{aligned}
\frac{d}{dx}\left(x^3+y^3\right) &= \frac{d}{dx}(6xy) \\
3x^2 + 3y^2\frac{dy}{dx} &= 6\left(y + x\frac{dy}{dx}\right) \\
(3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\
\frac{dy}{dx} &= \frac{6y-3x^2}{3y^2-6x}.
\end{aligned}$$

At the point $(0,0)$ this doesn't actually give us a useful answer; if you look at the picture in Figure 7.1, you see that there's not a clear tangent line there since the curve crosses itself.

In contrast, at the point $(3,3)$ we have that

$$\frac{dy}{dx} = \frac{18-27}{27-18} = -1$$

and the equation of the tangent line is $y - 3 = -(x-3)$.

- If $y\cos(x) = 1 + \sin(xy)$, then

$$\begin{aligned}
\frac{d}{dx}(y\cos(x)) &= \frac{d}{dx}(1+\sin(xy)) \\
\frac{dy}{dx}\cos(x) - y\sin(x) &= \cos(xy)\left(y + x\frac{dy}{dx}\right) \\
\frac{dy}{dx}(\cos(x) - x\cos(xy)) &= y\cos(xy) + y\sin(x) \\
\frac{dy}{dx} &= \frac{y\cos(xy)+y\sin(x)}{\cos(x)-x\cos(xy)}.
\end{aligned}$$

- If $\sqrt{xy} = 1 + x^2y$, then

$$\begin{aligned}
\frac{d}{dx}\sqrt{xy} &= \frac{d}{dx}\left(1+x^2y\right) \\
\frac{1}{2}(xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) &= 2xy + x^2\frac{dy}{dx} \\
\frac{dy}{dx}\left(x^2 - \frac{1}{2}x(xy)^{-1/2}\right) &= \frac{1}{2}(xy)^{-1/2}y - 2xy \\
\frac{dy}{dx} &= \frac{\frac{1}{2}(xy)^{-1/2}y - 2xy}{x^2 - \frac{1}{2}x(xy)^{-1/2}}.
\end{aligned}$$

Example 7.2. We can also compute second derivatives implicitly. If $9x^2 + y^2 = 9$ then we have

$$\begin{aligned}
18x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{9x}{y} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{9x}{y}\right) \\
&= -\frac{9y - 9x\frac{dy}{dx}}{y^2} \\
&= -\frac{9y - 9x(-\frac{9x}{y})}{y^2} \\
&= -\frac{9y + \frac{81x^2}{y}}{y^2}
\end{aligned}$$

We see that at the point $(0, 3)$ we have $y' = 0$ and $y'' = -3$. At the point $(\sqrt{5}/3, 2)$, then $y' = -\frac{3\sqrt{5}}{2}$ and $y'' = -\frac{18+\frac{45}{2}}{4}$.

Example 7.3. Find y'' if $x^6 + \sqrt[3]{y} = 1$. Then find the first and second derivatives at the point $(0, 1)$.

$$\begin{aligned}
6x^5 + \frac{1}{3}y^{-2/3}y' &= 0 \\
-18x^5y^{2/3} &= y' \\
-18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}y') &= y'' \\
-18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}(-18x^5y^{2/3})) &= y''
\end{aligned}$$

Thus at $(0, 1)$, we have $y' = 0$ and $y'' = 0$. So the tangent line to the curve is horizontal at the point $(0, 1)$.

We can also use implicit differentiation on relationships that apply to functions.

Example 7.4. Suppose we have some function f such that $8f(x) + x^2(f(x))^3 = 24$, and we want to find a linear approximation of f near $f(4) = 1$. (Say we've measured this experimentally and now want to understand or compute with the function). Then we have

$$\begin{aligned}
\frac{d}{dx}\left(8f(x)+x^2(f(x))^3\right) &= \frac{d}{dx}24 \\
8f'(x)+2x(f(x))^3+3x^2(f(x))^2f'(x) &= 0 \\
8f'(4)+2\cdot 4\cdot 1^3+3\cdot 4^2\cdot 1^2 f'(4) &= 0 \\
8f'(4)+8+48f'(4) &= 0
\end{aligned}$$

and thus $f'(4) = -1/7$.

This leaves us with a question, though. We know $f(1)$; can we figure out the value of f at other points?

From the work we just did, we can see the linear approximation is

$$f(x) \approx f'(4)(x-4)+f(4) = \frac{-1}{7}(x-4)+1.$$

Thus we compute

$$f(5) \approx \frac{-1}{7}(5-4)+1 = 1+\frac{-1}{7} = \frac{6}{7} \approx .857.$$

Checking Mathematica, we see that the actual solution is .879. So we were pretty close. But can we get closer?

7.2 Differential Equations

In the previous section we got the equation

$$8f'(x)+2x(f(x))^3+3x^2(f(x))^2f'(x) = 0.$$

An equation like this is called a *differential equation* because it relates a function to its derivative. Differential equations come up very often in science, economics, and other fields that use mathematical modeling, because it is often easy to state a natural law or modelling assumption in this form.

For a simple example, consider the phrase “acceleration is proportional to force.” Recall that acceleration is the second derivative of position. If force is itself a function of position, this translates to a differential equation, relating $f''(x)$ to $f(x)$.

Example 7.5. Hooke’s law tells us that the force a spring exerts is proportional to the displacement of the spring; that is, for any given spring there is some constant k such that $F(t) = -kf(t)$. Since $F(t) = ma(t) = mf''(t)$, this gives us the differential equation $mf''(t) = -kf(t)$ or

$$f''(t) = -\frac{k}{m}f(t).$$

For simplicity let's assume $k = m$ so we have $f''(t) = -f(t)$.

Can we find a solution for this? We can start with the really silly or "trivial" solution. If the spring starts at neutral, it will never move, so we'd expect $f(t) = 0$. And indeed it is: $0'' = 0 = -0$, so the funcion $f(t) = 0$ is a solution to this differential equation.

Can we find a solution that involves any motion at all? We're looking for a function where $f''(t) = -f(t)$. And we actually know two of these: $f(t) = \sin(t)$ and $f(t) = \cos(t)$ both satisfy this differential equation. And this is why the equation for "simple harmonic motion" is built up out of sin and cos functions.

There are many different solutions we can use; for example, $3\sin(t) + 5\cos(t) = 17$ is a solution to this differential equation. To pick out the specific solution we need to know "initial conditions" that tell us the starting position.

There is a rich and powerful theory for solving differential equations. We will not be investigating it in this course, but there are a few questions we can address.

Example 7.6. Suppose $f(x) = ax^2 + bx + c$ satisfies $f'(x) = 4x + 3$ and $f(0) = 0$. What is $f(x)$?

We compute $f'(x) = 2ax + b = 4x + 3$, so we have $a = 2, b = 3$. Then $f(x) = 2x^2 + 3x + c$. Since $f(0) = 0$ we have $c = 0$, and thus $f(x) = 2x^2 + 3x$.

Example 7.7. Suppose $f(x) = ax^2 + bx + c$ is a polynomial, and we have $f(0) = 0, f'(0) = 1, f''(0) = 2$. What can we say about $f(x)$?

We see that $f(0) = c = 0$, $f'(x) = 2ax + b$ so $f'(0) = b = 1$, and $f''(x) = 2a$ so $f''(0) = 2a = 2$. Thus $a = b = 1$ and $c = 0$, so $f(x) = x^2 + x$.

Example 7.8. Suppose $g(x) = ax^2 + bx + c$ is a polynomial, with $g(1) = 2, g'(2) = 3, g''(3) = 4$. What can we say about g?

We have $g(1) = a + b + c$. $g'(x) = 2ax + b$ so $g'(2) = 4a + b = 3$, and $g''(x) = 2a$ so $g''(3) = 2a = 4$. Thus we have $a = 2$. Going back to g' we see that $8 + b = 3$ so $b = -5$. Then plugging into g we have $2 - 5 + c = 2$ so $c = 5$. Thus $g(x) = x^2 - 5x + 5$.

Example 7.9. Confirm that $f(x) = x^2 + x + 1$ satisfies $2f(x) - xf'(x) = x + 2$.

We compute $f'(x) = 2x + 1$, so $2f(x) - xf'(x) = 2x^2 + 2x + 2 - (2x^2 + x) = x + 2$.

7.3 Euler's Method

We cannot develop a general method of *exactly* solving even simple differential equations in this course, since this requires integrals. But we can come up with approximate solutions. And sometimes approximate solutions are the best that anyone can do!

Example 7.10. Let's consider about the simplest possible non-trivial differential equation: $f'(x) = f(x)$. And let's add in the information that $f'(0) = 1$. What can we say about the values of this function?

Let's start by approximating $f(1)$. From our differential equation we know that $f'(0) = f(0) = 1$, so by linear approximation we have

$$f(1) \approx f'(0)(1-0) + f(0) = 1 + 1 = 2.$$

But of course this isn't an exact answer. Can we be more precise?

Recall our approximations get less and less accurate as we get farther away from our base point, because the derivative keeps changing. We can improve our accuracy by stopping halfway through to correct our estimate of the rate of change.

$$\begin{aligned} f(.5) &\approx f'(0)(.5-0) + f(0) = 1 \cdot .5 + 1 = 1.5 \\ f(1) &\approx f'(.5)(1-.5) + f(.5) \approx 1.5(.5) + 1.5 = 2.25. \end{aligned}$$

We can always get more precision by using more steps.

$$\begin{aligned} f(1/4) &\approx f'(0)(1/4-0) + f(0) = 1(1/4) + 1 = 5/4 \\ f(1/2) &\approx f'(1/4)(1/2-.1/4) + f(1/4) \approx 5/4(1/4) + 5/4 = 25/16 \\ f(3/4) &\approx f'(1/2)(3/4-1/2) + f(1/2) \approx \frac{25}{16}(1/4) + \frac{25}{16} = \frac{125}{64} \\ f(1) &\approx f'(3/4)(1-3/4) + f(3/4) \approx \frac{125}{64} \cdot \frac{1}{4} + \frac{125}{64} = \frac{625}{256} \approx 2.44. \end{aligned}$$

We will see in Section 8 that the exact solution to this problem is $e \approx 2.71828$.

(We will make a note to recall later that with one step, we had $(1+1)^1$; with two steps, we had $(3/2)^2$; and with four steps we had $(5/4)^4$. We can see that in general, with n steps we will have $((n+1)/n)^n$ as our approximation).

This approach to approximating solutions to a differential equation is known as *Euler's Method*:

1. Pick a step size δ.
2. Start with a base point whose value is known: $f(x_0) = y_0$.
3. Use the differential equation to compute $f'(x_0)$.
4. Use a linear approximation to approximate $f(x_0 + \delta)$.

5. Take this as your new base point, compute $f'(x_0+\delta)$, and then approximate $f(x_0+2\delta)$.

6. Repeat until you have approximated your desired output.

Example 7.11. Suppose $f'(t) = f(t) - f(t)^2/2$, and $f(0) = 1$. Let us approximate $f(3)$ using 3 steps, for a step size of 1.

$$\begin{aligned}
f(1) &\approx f'(0)(1-0) + f(0) = (1 - 1^2/2)(1) + 1 = 3/2. \\
f(2) &\approx f'(1)(2-1) + f(1) \approx \left(\frac{3}{2} - \frac{\left(\frac{3}{2}\right)^2}{2}\right)(1) + \frac{3}{2} = \frac{3}{8} + \frac{3}{2} = \frac{15}{8}. \\
f(3) &\approx f'(2)(3-2) + f(2) \approx \left(\frac{15}{8} - \frac{\left(\frac{15}{8}\right)^2}{2}\right)(1) + \frac{15}{8} = \frac{15}{128} + \frac{15}{8} = \frac{255}{128}.
\end{aligned}$$

Thus we estimate $f(3) \approx 1.99$.

Example 7.12. Suppose $f'(x) = x - f(x)$ and $f(1) = 3$. Let's approximate $f(2)$ with a step size of $1/4$. We have

$$\begin{aligned}
f(5/4) &\approx f'(1)(1/4) + f(1) = (1-3)(1/4) + 3 = \frac{5}{2} \\
f(3/2) &\approx f'(5/4)(1/4) + f(5/4) \approx \left(\frac{5}{4} - \frac{5}{2}\right)\frac{1}{4} + \frac{5}{2} = \frac{35}{16} \\
f(7/4) &\approx f'(3/2)(1/4) + f(3/2) \approx \left(\frac{3}{2} - \frac{35}{16}\right)\frac{1}{4} + \frac{35}{16} = \frac{129}{64} \\
f(2) &\approx f'(7/4)(1/4) + f(7/4) \approx \left(\frac{7}{4} - \frac{129}{64}\right)\frac{1}{4} + \frac{129}{64} = \frac{499}{256}.
\end{aligned}$$

Thus we estimate $f(2) \approx \frac{499}{256} \approx 1.95$.

7.4 Word Problems and Related Rates

Sometimes we have word problems that require us to translate verbal information into equations, and then solve the problem.

Checklist of steps for solving word problems:

1. Draw a picture.

2. Think about what you expect the answer to look like. What is physically plausible?

3. Create notation, choose variable names, and label your picture.

   (a) Write down all the information you were given in the problem.

(b) Write down the question in your notation.

4. Write down equations that relate the variables you have.

5. Abstractly: "solve the problem." Concretely differentiate your equation.

6. Plug in values and read off the answer.

7. Do a sanity check. Does you answer make sense? Are you running at hundreds of miles an hour, or driving a car twenty gallons per mile to the east?

Example 7.13. Suppose one car drives north at 40 mph, and an hour later another starts driving west from the same place at 60 mph. After a second hour, how quicly is the distance between them increasing?

Write a for the distance the first car has traveled, and b for the distance the second car has traveled. We have that $a = 80, b = 60, a' = 40, b' = 60$. If the distance between the cars is d then after two hours, $d = 100$, and we have

$$\begin{aligned} d^2 &= a^2 + b^2 \\ 2dd' &= 2aa' + 2bb' \\ 2 \cdot 100 \cdot d' &= 2 \cdot 80 \cdot 40 + 2 \cdot 60 \cdot 60 \\ d' &= \frac{3200 + 3600}{100} = 68, \end{aligned}$$

so the distance between the cars is increasing at 68 mph. This seems reasonable because the cars are traveling at 40 mph and 60 mph.

Example 7.14. A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end sliding out when the top is 12 feet from the ground?

Let h be the height of the ladder on the wall, and b be the distance of the foot of the ladder from the wall. Then $h = 12, h' = -1$, and $b = \sqrt{400 - 144} = 16$. We have

$$\begin{aligned} h^2 + b^2 &= 400 \\ 2hh' + 2bb' &= 0 \\ 2 \cdot 12 \cdot (-1) + 2 \cdot 16 \cdot b' &= 0 \\ b' &= \frac{24}{32} = 3/4 \end{aligned}$$

so the foot of the ladder is sliding away from the wall at 3/4 ft/s. Again, the direction of the sliding is correct (away from the wall), and the number seems plausible.

Example 7.15. A spherical balloon is inflating at 12 cm^3 per second. How quickly is the radius increasing when the radius is 3 cm?

A sphere has volume $V = \frac{4}{3}\pi r^3$. We have $V' = 12$ and $r = 3$. We compute

$$\begin{aligned} V' &= 4\pi r^2 r' \\ 12 &= 4\pi(3)^2 r' \\ r' &= \frac{1}{3\pi} \end{aligned}$$

So the radius is increasing by $1/3\pi$ cm per second.

Example 7.16. A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

We have $l = 5, w = 7, l' = 1, w' = 2$, and $A = lw$. Taking a derivative gives us $A' = lw' + wl' = 5 \cdot 2 + 7 \cdot 1 = 17$ square inches per second.

Example 7.17. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of $2\text{m}^3/\text{min}$. How fast is the water rising when the water is 3 m tall?

Let h be the current height of the water, r the current radius, and V the current volume of water. We know that $h = 3$, and by similar triangles we see that $\frac{h}{r} = \frac{4}{2}$ and thus $r = h/2$. We know that $V' = 2$, and the volume formula for a cone gives us $V = \frac{1}{3}\pi r^2 h$. We compute

$$\begin{aligned} V &= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi \frac{h^3}{4} \\ V' &= \frac{\pi}{4} h^2 h' \\ 2 &= \frac{\pi}{4} 3^2 h' \\ \frac{8}{9\pi} &= h', \end{aligned}$$

so the water level is rising at $\frac{8}{9\pi}$ meters per minute.

Example 7.18. A street light is mounted at the top of a 15-foot-tall pole. A six-foot-tall man walks straight away from the pole at 5 feet per second. How fast is the tip of his shadow moving when he is forty feet from the pole?

Let d be the distance of the man from the pole, and L be the distance from the pole to the tip of his shadow. We have $d' = 5$ and we set up a similar triangles equation.

$$\frac{15}{L} = \frac{6}{L-d} \qquad 6L = 15L - 15d$$
$$9L = 15d \qquad d = \frac{3}{5}L$$
$$d' = \frac{3}{5}L' \qquad 5 = \frac{3}{5}L'$$

and thus the tip of his shadow is moving at $\frac{25}{3}$ feet per second.

Example 7.19. A lighthouse is located three kilometers away from the nearest point P on shore, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline 1 kilometer from P?

Let's say the angle of the light away from P is θ, and the distance from P is d. Then we have $d = 1$ and $\theta' = 8\pi$ (in radians per minute). We also have the relationship that $\tan\theta = \frac{d}{3}$.

Taking the derivative gives us $\sec^2(\theta)\cdot\theta' = d'/3$. We need to work out $\sec^2(\theta)$, but looking at our triangle we see that the adjacent side is length 3 and the hypotenuse is length $\sqrt{10}$ (by the Pythagorean theorem), so we have $\sec^2(\theta) = (\sqrt{10}/3)^2 = 10/9$.

Thus we have $d' = 3\sec^2(\theta)\cdot 8\pi = \frac{80\pi}{3}$ kilometers per second.

Example 7.20. A kite is flying 100 feet over the ground, moving horizontally at 8 ft/s. At what rate is the angle between the string and the ground decreasing when 200ft of string is let out?

Call the distance between the kite-holder and the kite d and the angle between the string and the ground θ. When the length of string is 200 then $d = \sqrt{200^2 - 100^2} = 100\sqrt{3}$. We have that $d' = 8$ (since the angle is decreasing, the kite must be getting farther away). And finally we have the relationship $\tan\theta = \frac{100}{d}$ by the definition of tan in terms of triangles. Then we have

$$\begin{aligned}\tan\theta &= 100d^{-1}\\ \sec^2(\theta)\theta' &= -100d^{-2}d'\\ \theta' &= \frac{-100\cdot 8\cos^2(\theta)}{d^2}.\end{aligned}$$

We see that $\cos(\theta) = \frac{100\sqrt{3}}{200} = \sqrt{3}2$, so we have

$$\theta' = \frac{-100\cdot 8\cdot 3/4}{(100\sqrt{3})^2} = -\frac{8}{100\cdot 4} = \frac{-1}{50}.$$

So the angle between the string and the ground is decreasing at a rate of 1/50 per second. (Note: radians are unitless!)