Math 114 Fall 2018
Calculus I HW 2 Solutions
Due Wednesday, September 12

1. Find, with proof, $\lim_{x \to 3} 4x$.

   Solution: We guess $4 \cdot 3 = 12$.

   Let $\epsilon > 0$ and let $\delta = \underline{\epsilon/4}$. Then if $|x - 3| < \delta$, we have

   $$|4x - 12| = |4(x - 3)| = 4|x - 3| < 4\delta = 4\epsilon/4 = \epsilon.$$

2. Find, with proof, $\lim_{x \to 2} (x + 1)^2$.

   Solution: We guess $(2 + 1)^2 = 9$.

   Let $\epsilon > 0$ and let $\delta < \underline{\epsilon/7, 1}$. Then if $|x - 2| < \delta$ we have

   $$\begin{aligned} |(x+1)^2 - 9| &= |x^2 + 2x + 1 - 9| = |x^2 + 2x - 8| = |(x^2 - 4) + 2(x - 2)| \\ &\leq |x - 2| \cdot |x + 2| + 2|x - 2| \leq |x + 2|\delta + 2\delta \\ &= \delta(2 + |x - 2 + 4|) \leq \delta(2 + |x - 2| + 4) \leq \delta(7) < \epsilon. \end{aligned}$$

   Alternate Solutions: We can compute $|(x + 1)^2 - 9| < \delta(6 + \delta)$ and then solve the quadratic equation $\delta^2 + 6\delta - \epsilon = 0$ for δ, giving us $\delta = \frac{-6 \pm \sqrt{36 + 4\epsilon}}{2}$. Thus if $x < \delta = \frac{-6 + \sqrt{36 + 4\epsilon}}{2} = -3 + \sqrt{9 + \epsilon}$ then $|f(x) - 9| < \epsilon$.

   Alternatively again, we can observe

   $$|(x + 1)^2 - 9| = |x^2 + 2x - 8| = |x + 4| \cdot |x - 2| < |x + 4|\delta \leq \delta(|x - 2| + 6)$$

   which then follows through into either of the previous solutions.

3. Find, with proof, $\lim_{x \to 1} x^2$.

   Solution: We guess $1^2 = 1$.

   Let $\epsilon > 0$ and set $\delta < \underline{\epsilon/3, 1}$. Then if $|x - 1| < \delta$ we compute

   $$\begin{aligned} |x^2 - 1| &= |x - 1| \cdot |x + 1| = |x - 1| \cdot |x - 1 + 2| \leq |x - 1|(|x - 1| + 2) \\ &< \delta(1 + 2) < 3\epsilon/3 = \epsilon. \end{aligned}$$

Alternate Solutions: We can observe that $|x-1|(|x-1|+2) < \delta(\delta+2)$ and solve $\delta^2 + 2\delta - \epsilon = 0$ for δ, giving $\delta = \frac{-2\pm\sqrt{4+4\epsilon}}{2} = -1 \pm \sqrt{1+\epsilon}$. Then we observe that if $x < \delta = -1+\sqrt{1+\epsilon}$ then $|x^2-1| < \epsilon$.

4. Find, with proof, $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$.

   Solution: We guess that we can cancel out an $x-3$, and thus get $3+3=6$.

   Let $\epsilon > 0$ and set $\delta = \underline{\epsilon}$. Then if $0 < |x+3| < \delta$, we compute

$$\begin{aligned}\left|\frac{x^2-9}{x-3} - 6\right| &= \left|\frac{(x-3)(x+3)}{x-3} - 6\right| = |(x+3)-6| \\ &= |x-3| < \delta = \epsilon.\end{aligned}$$

5. $\star$ Find, with proof, $\lim_{x\to 2} \dfrac{1}{x-1}$.

   Solution: We guess $1/(2-1) = 1$.

   Let $\epsilon > 0$ and set $\delta < \underline{\epsilon/2, 1/2}$. Then if $|x-2| < \delta$, we compute

$$\left|\frac{1}{x-1} - 1\right| = \left|\frac{1-1(x-1)}{x-1}\right| = \left|\frac{2-x}{x-1}\right| = \frac{|x-2|}{|x-1|}$$

   But we know that

$$|x-1| = |x-2+1| = |1-(2-x)| \geq 1 - |x-2| \geq 1-\delta \geq 1/2$$

   and thus

$$\frac{|x-2|}{|x-1|} \leq \frac{\delta}{1-\delta} < \frac{\epsilon/2}{1/2} = \epsilon.$$

6. $(\star)$ Find (with proof) $\lim_{x\to 5} \frac{1}{x-4}$.

   Solution: Let $\epsilon > 0$ and let $\delta \leq \underline{1/2, \epsilon/2}$. Then if $|x-5| < \delta$, we compute

$$\begin{aligned}\left|\frac{1}{x-4} - 1\right| &= \left|\frac{1-(x-4)}{x-4}\right| \\ &= \frac{|5-x|}{|x-4|} < \frac{\delta}{|x-4|}.\end{aligned}$$

   Since the denominator is positive, we need to make the denominator big, and so use the reverse triangle inequality. Then we compute

$$|x-4| = |(x-5)+1| = |1-(5-x)| \geq 1 - |5-x| > 1-\delta \geq 1/2$$

   after we set $\delta \leq 1/2$. Thus

$$\left|\frac{1}{x-4} - 1\right| < \frac{\delta}{|x-4|} < \frac{\delta}{1/2} = 2\delta < \epsilon.$$

7. Let

$$f(x) = \begin{cases} 1 & x < 2 \\ 2 & x = 2 \\ 3 & x > 2 \end{cases}$$

What is $f(2)$? Prove that $\lim_{x\to 2} f(x)$ does not exist.

Solution: $f(2) = 2$ by definition of the function.

Suppose $\lim_{x\to 2} f(x) = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x - 2| < \delta$ then $|f(x) - L| < \epsilon = 1$.

Let $x_1 = 2 + \delta/2$, so that $|x_1 - 2| = \delta/2 < \delta$; then $f(x_1) = 3$, and thus $1 > |f(x_1) - L| = |3 - L|$.

Let $x_2 = 2 - \delta/2$, so that $|x_2 - 2| = \delta/2 < \delta$; then $f(x_2) = 1$ and $1 > |f(x_2) - L| = |1 - L| = |L - 1|$. (We flip the sign inside the absolute value to make the next step easier, when we would like the Ls to cancel).

Adding these two inequalities and applying the triangle inequality gives us

$$1 + 1 > |L - 1| + |3 - L| \geq |(L - 1) + (3 - L)| = 2,$$

but this is a contradiction, so the limit must not exist.