Math 114 Fall 2018
Calculus I HW 3 Solutions
Due Wednesday, September 20

1. Let

$$j(x) = \begin{cases} 3x - 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$$

Show that $\lim_{x \to 3} j(x) = 7$.

Solution: Let $\epsilon > 0$ and fix $\delta \leq \underline{3, \epsilon/2}$. Then if $0 < |x - 3| < \delta$ we know that $x > 0$, and so we have

$$|j(x) - 7| = |2x + 1 - 7| = |2x - 6| = 2|x - 3| < 2\delta \leq \epsilon.$$

2. ($\star$)For the same function j, show that $\lim_{x \to 0} j(x)$ does not exist.

Solution: Suppose $\lim_{x \to 0} j(x) = L$. Fix $\epsilon = \underline{1}$ and suppose $\delta > 0$. Pick $x_1 = \delta/2$ and $x_2 = -\delta/2$.

Then since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |2\delta/2 + 1 - L| = |\delta + 1 - L|.$$

Similarly, since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = | - 3\delta/2 - 1 - L| = |L + 1 + 3\delta/2|.$$

Adding these equations gives

$$\begin{aligned} 2\epsilon &> |\delta + 1 - L| + |L + 1 + 3\delta/2| \\ &\geq |\delta + 1 - L + L + 1 + 3\delta/2| = |5\delta/2 + 2| \\ &= 2 + 5\delta/2. \end{aligned}$$

Since $\epsilon = 1$ this gives us $2 > 2 + 5\delta/2$ and thus $0 > \delta$ which is a contradiction. So no such limit exists.

3. ($\star$) Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution: Suppose $\lim_{x\to 0} \frac{|x|}{x} = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x| < \delta$ then $\left|\frac{|x|}{x} - L\right| < \epsilon = 1$.

Let $x_1 = \delta/2 < \delta$; then

$$1 = \epsilon > \left|\frac{|x_1|}{x_1} - L\right| = \left|\frac{|\delta/2|}{\delta/2} - L\right| = |1 - L|.$$

Let $x_2 = -\delta/2$ so that $|x_2| = \delta/2 < \delta$; then

$$1 = \epsilon > \left|\frac{|x_2|}{x_2} - L\right| = \left|\frac{|-\delta/2|}{-\delta/2} - L\right| = |-1 - L| = |L + 1|.$$

Then adding the two inequalities and using the triangle inequality gives us

$$2 = 1 + 1 > |L + 1| + |1 - L| \geq |(L + 1) + (1 - L)| = 2,$$

but this is a contradiction, so the limit must not exist.

4. From the definition, prove that $\lim_{x\to 2} \frac{1}{x-2} = \pm\infty$.

Solution: Let $N > 0$ and set $\delta = \underline{1/N}$. Then if $0 < |x - 2| < \delta$, then

$$\left|\frac{1}{x-2}\right| = \frac{1}{|x-2|} > \frac{1}{\delta} = \frac{1}{1/N} = N.$$

Thus $\lim_{x\to 2} \frac{1}{x-2} = \pm\infty$.

5. From the definition, prove that $\lim_{x\to -1} \frac{4}{(x+1)^2} = +\infty$.

Solution: Let $N > 0$ and set $\delta = \underline{2/\sqrt{N}}$. Then if $0 < |x + 1| < \delta$, then

$$\frac{4}{(x+1)^2} > \frac{4}{\delta^2} \geq \frac{4}{(2/\sqrt{N})^2} = N.$$

6. From the definition, prove that $\lim_{x\to 3} \frac{-2}{|x-3|} = -\infty$.

Solution: Let $N > 0$ and set $\delta = \underline{2/N}$. Then if $0 < |x - 3| < \delta$, then

$$\begin{aligned} |x-3| &< \delta \\ \frac{1}{|x-3|} &> \frac{1}{\delta} \\ \frac{-2}{|x-3|} &< \frac{-2}{\delta} = \frac{-2}{2/N} = -N. \end{aligned}$$

7. Let a and c be any constants. From the ϵ-δ definition, prove that $\lim_{x\to a} c = c$.

Solution: Let $\epsilon > 0$ and let $\delta = 1$. Then if $0 < |x - a| < \delta$, we compute

$$|f(x) - c| = |c - c| = 0 < \epsilon.$$