Lab 2 Tuesday September 4

Visualizing limits

Recall from last week that we can plot a function `f[x]`, on the domain $[a, b]$, with the command `Plot[f[x],{x,a,b}]`

Our goal for today is to represent limits graphically. Recall that for a limit $\lim_{x\to a} f(x) = L$ to exist, for any error margin ϵ we need to find a distance δ so that if x is within δ of a, then $f(x)$ is always within ϵ of L.

We'll start with an example. Let's consider the function x^2.

1. Plot the function x^2 around the point $a = 0$ with the command `Plot[x^2, {x,0-2,0+2}]` Guess/remember $\lim_{x\to 0} x^2$.

2. For now, let's set the error margin to $\epsilon = 1$. We can plot lines at $0 \pm \epsilon$ by running the command `Plot[{x^2,0-1,0+1},{x,0-2,0+2}]` so that our error band is the area between the two lines. Based on this picture, if our input is between -2 and 2, will our output be within our error margin? What is the δ we are using for this picture—the horizontal distance we allow from zero—and is it close enough that our outputs are all inside the error margin?

3. What does δ need to be to make our output land in our error margin? Plot another graph with the same error margin but a smaller domain, so that all your outputs are within the error margin.

4. If we use an error margin of $\epsilon = 1/4$, what δ do we need? Plot the corresponding graph.

5. Plot another graph for $\epsilon = 1/10$.

Bonus: Come up with a formula for what δ needs to be, in terms of ϵ. We'll discuss this in detail in tomorrow's class. Then use the following code:

```
epsilon = 1

a=0

L=0

delta = Sqrt[epsilon]

Plot[{L-epsilon,x^2,L+epsilon},{x,a-delta,a+delta}
```

Run this code with several different values of ϵ. Does it work every time?

I will also demonstrate for $f(x) = 3x, a = 1$ and for $f(x) = 1/x, a = 4$ I will show what happens if we guess the wrong limit, as if we try to prove that $\lim_{x\to 4} 1/x = 1$.

The Most Important Code Formula

```
Plot[{f[x],L-epsilon, L + epsilon},{x, a - delta, a + delta}]
```

Here `f[x]` is the function, `a` is the point at which you're calculating the limit, `L` is the limit, `epsilion` is the allowable error margin, and `delta` is the allowable horizontal distance that achives this error margin.

Exercises

Below there is a list of functions f paired with numbers a. For each item of the list:

1. Plot a graph of f centered at the point a.
2. Use this graph to estimate $L = \lim_{x \to a} f(x)$.
3. Plot a graph with an error margin given by $\epsilon = 1$. What δ do we need to make all outputs fall within ϵ of L?
4. Do the same with $\epsilon = 1/2, 1/10, 1/100$.

(a) $f(x) = 2x, a = -2$

(b) $f(x) = 1/x, a = 10$

(c) $f(x) = 3, a = 0$

(d) `f[x_]:=Abs[x]/x`, $a = 0$.

(e) $f(x) = x^2 + 3, a = 0$

(f) $f(x) = \frac{x^2-4}{x-2}, a = 2$

(g) $f(x) = x^3 + x, a = 1$

(h) $f(x) = \frac{x-1}{x^2-1}, a = 1$.

Bonus: $f(x) = \sin(x), a = 0$

Extra Visualization: Two Deltas

Now let's look at the limit of $f(x) = x^2$ as x approaches 3. If we take $\epsilon = 1$, what delta will work? We find that $\delta = 1/7$ works, with `Plot[{x^2,9-1, 9+1},{x,3-1/7,3+1/7}]`. So we might guess that $\delta = \epsilon/7$ will always work. And indeed, if we try this with $\epsilon = 1/10$ or $\epsilon = 1/100$ everything will still work fine. But what happens if we take $\epsilon = 10$? Run the command `Plot[{x^2,9-10, 9+10},{x,3-10/7,3+10/7}]` and see what happens.

Normally we think about limits as checking what happens when we "zoom in". And this means we think of ϵ as being small. But by our definition, the limit needs to work for *every* ϵ, not just every small ϵ.

Fortunately, this isn't really too much of a problem. Can you think of a rule we can make for δ that will work for every ϵ, no matter how big?