

Lab 5

Tuesday October 2

Small Angle Approximation

1. `Plot[{Sin[x], x}, {x, -1, 1}]`
2. `Plot[{Sin[5x], 5x}, {x, -1, 1}]`
3. `Plot[{Sin[x^2-4], x^2-4}, {x, 1, 3}]`
4. `Plot[{Sin[Tan[x]], Tan[x]}, {x, -1, 1}]`

Filling in functions

We can always take a function whose domain is not all reals and extend it to a function whose domain is all real numbers by simply assigning values to all the points that aren't already covered. You can think of this as defining a piecewise function where one of the pieces is just a single point. We often want to extend a given function to be a *continuous* function on all reals; however, this is not always possible.

Go to the course webpage and download the file called “plot piecewise file”. If the computer asks to open it with Mathematica, do that; if not, copy all the text (use `ctrl+a`) and then paste it into an empty Mathematica notebook, and when asked if you want to interpret the text, click “yes”. Click anywhere in the giant block of code and hit `shift+enter`; this gives you the new function `PlotPiecewise` which we will be using for the rest of this lab.

For each of the following functions, do the following steps:

1. Figure out the domain, and define the function `f[x]` in mathematica.
 2. Plot the function using the `PlotPiecewise` command from the notebook I posted on Moodle, and the command `PlotPiecewise[f[x], {x, -3, 3}]`
Use the domain given in the problem.
 3. Can you assign values to the function at the holes so that the function is “nice” or “connected” (continuous)? (“No” is a possible answer here, but think about why).
 4. If you can, define a new function `fFixed[x]` that has the holes filled in, using the `Piecewise` command. E.g. if there is a hole at $x = 2$ and you think the value “should” be 5, enter `fFixed[x_] := Piecewise[{{f[x], x != 2}, {5, x == 2}}`
(note we use `!=` for “not equals” and `==` for “equals”)
 5. Plot your new function in `PlotPiecewise` to see if you have successfully filled the holes.
`PlotPiecewise[fFixed[x], {x, -3, 3}]`
- (a) `f[x_] := (x^2-1)/(x-1)` on $(-2, 2)$. (e) `j[x_] := 2 Abs[x+1]/(2x+2)` on $(-2, 2)$.
- (b) `g[x_] := 1/(x+1)` on $(-2, 2)$. (f) `m[x_] := Sin[x-1]/(x-1)` on $(0, 2)$.
- (c) `h[x_] := (x+1)^2/(x^2-1)` on $(-2, 2)$. (g) `n[x_] := (x^3-8)/(x^2-4)` on $(-4, 4)$.
- (d) `i[x_] := (x^2-2x+1)/(x-1)` on $(-2, 2)$. (h) `o[x_] := (x^2-x)/(x^3-3x^2+2x)` on $(-2, 4)$.

Secant and Tangent Line Exercises

In this lab, we will see the process of approximating a curve via a secant line. At the bottom of this sheet, there is a list of functions and of x -values. **For each pair f, a you should do the following steps;** I show you code for the pair $f[x_] := x^3/2-x$ and $a=0$.

1. Plot the graph of f , and imagine what the tangent line will look at near a .

```
f[x_] := x^3/2-x
Plot[f[x], {x, -2, 2}]
```

2. Pick some points close to a , such as $a+1, a+.1, a-1, a-.1$, and so forth. On one graph, plot the function f , and the line connecting two points on the graph near a (using the two-point equation. The equation for this line in Mathematica code is $(x-a)(f[b]-f[a])/(b-a)+f[a]$).

I pick the points $-1, -.5, -.1, -.01, 1, .5, .1, .01$; I'll show code for lines connecting the first two of these points to $(0, f(0))$ below.

```
Plot[{f[x], (x-0) (f[-1]-f[0])/(-1-0) + f[0]}, {x, -2, 2}]
Plot[{f[x], (x-0) (f[-.5]-f[0])/(-.5-0) + f[0]}, {x, -2, 2}]
```

When you follow these steps for each function, make sure you **try some points on the right and some points on the left!** Usually the same thing will happen, but not always. What happens as the points get closer together?

3. Compute the slopes of each of the lines you plotted. It's easiest to do this with a `Table` command.

```
Table[(f[b] - f[0])/(b-0), {b, {-1, -.5, -.1, -.01, -.001, 1, .5, .1, .01, .001}}]
```

What is the limit as your points get closer to a ?

4. Use Mathematica to compute the "derivative" at a , which should be the same as the limit you just computed. There are two ways to do this; one of them is the obvious way. Try each at least once.

```
f'[0] or D[f[x], x]/.x->0
```

5. Plot on the same graph f and the tangent line to f at a . Remember the point slope formula

$$y = m(x - a) + f(a).$$

```
Plot[{f[x], f'[0](x-0) + f[0]}, {x, -1, 1}]
```

Do all of the previous steps for each of the following functions and points a :

- | | |
|---|--|
| (a) $f[x_] := x^3/2 - x$ and $a = 2$ | (f) <code>CubeRoot[x]</code> and $a=1$ |
| (b) $g[x_] := x^5+1$ and $a = 2$ | (g) <code>CubeRoot[x]</code> and $a=0$ |
| (c) $h[x_] := x^3+x^2-1$ and $a = -1$ | (h) $m[x_] := \text{CubeRoot}[x^2]$ and $a = -1$ |
| (d) <code>Sin[x]</code> and $a = 0$ | (i) $m[x_] := \text{CubeRoot}[x^2]$ and $a = 0$ |
| (e) $k[x_] := \text{Piecewise}[\{\{-1, x < 0\}, \{1, x \geq 0\}\}]$ | (j) <code>Abs[x]</code> and $a = 0$ |

Secant Lines and Tangent Lines

When we have a graph of a straight line, we can talk about its slope, and understand important things about how the function behaves (and in particular, how quickly the output changes in response to changes in the input). If our function is not a straight line, then it does not have a slope; but we'd like to talk about what slope it "ought to" have anyway. (We will discuss this more in class soon).

First we recall some facts about lines. There are a few line equations we may want to use:

$$\begin{array}{ll} y = mx + b & \text{Slope-Intercept Formula} \\ y - y_0 = m(x - x_0) & \text{Point-Slope Formula} \\ y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) & \text{Point-Point Formula} \end{array}$$

In order to write down the equation of a line, we need two pieces of information: either we need two points, or we need one point and also the slope.

Definition 0.1. A line that touches a curve at one point without crossing it is *tangent* to the curve at that point, and we call such a line a *tangent line* (from Latin *tangere* "to touch".)

A line crossing a curve in two points is called a *secant* line. (from Latin *secare* "to cut").

Just as the tangent of an angle is the length of a tangent line segment, the secant of an angle is the length of a secant line segment.

We calculate what the slope of a function at a point "ought to be" by drawing lines that approximate our function. A secant line through two points of the function graph will be similar to the function; as the points get closer together, the line will better approximate the bit of the function between them. In the limit as the points get arbitrarily close together, the secant line becomes a very good approximation of our function at a single point—and also becomes the tangent line at that point.

(Why not just start with the tangent line? It's difficult to see the equation for a tangent line. To write the equation for a line we either need two points, or a point and the slope—and if we knew the slope we wouldn't need to do any of this to begin with, so we really need two points. Since the secant line intersects two points, we can use the point-point formula for a line to get an equation.)