

## Lab 6

Tuesday October 23

## Approximating Functions with Derivatives

Last lab, we drew secant lines, which are lines that intersect the graph of a function in (at least) two points; we may recall that by rearranging some information, we can write

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a).$$

As  $x$  approaches  $a$ , this becomes closer to being a tangent line, and the slope term becomes closer to  $f'(a)$ . Thus we can get a decent approximation, if  $x$  and  $a$  are close, by replacing this difference quotient with the derivative:

$$f(x) \approx f'(a)(x - a) + f(a).$$

Another approach to this same idea is to think geometrically, about the tangent line. Last week in lab we drew tangent lines, and saw that the tangent line to a function looks very similar to that function close up. That is, the tangent line to  $f$  at  $a$  is the line that most resembles  $f$  near the point  $a$ . Thus if we write the equation to the tangent line

$$y = f'(a)(x - a) + f(a)$$

then we get the line which most closely approximates the function  $f$ ; we can plug  $x$  values into this formula to find approximate calculations of  $f(x)$ , with the approximation improving if  $x$  is closer to  $a$ .

If we want to compute  $f[b]$  for some specific and awkward value  $b$ , we can:

- (a) Find a value  $a$  which is close to  $b$ , such that  $f[a]$  is easy to compute.

e.g. if  $f(x) = \sqrt{x}$  and  $b = 5$  we may pick  $a = 4$ .

- (b) Calculate the derivative at the point  $a$ :  $f'[a]$ .

If  $f(x) = \sqrt{x}$  and  $b = a$  then  $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{4}$ .

- (c) Find an equation  $y = m(x - a) + f(a)$  for the line tangent to  $f$  at  $a$ . (Check your work by plotting  $f$  and your line on the same plot in Mathematica). What is the  $y$  value given by this line when the  $x$ -coordinate is  $b$ ?

We know the slope of the tangent line at  $a$  is  $\frac{1}{4}$ , and  $f(a) = 2$ , so our line is  $y = \frac{1}{4}(x - 4) + 2$ .

We can plot this in Mathematica with

`Plot[{Sqrt[x], (1/4) (x-4) + 2}, {x, 2, 6}]` or

`Plot[{Sqrt[x], Sqrt'[4] (x-4) + Sqrt[4]}, {x, 2, 6}]`

When we plug in 5 (our  $b$  value) for  $x$  we get  $y = \frac{1}{4}(5 - 4) + 2 = \frac{1}{4} + 2 = \frac{9}{4}$ . Thus we guess  $\sqrt{5} \approx \frac{9}{4}$ .

- (d) Check your answer from (c) by evaluating  $f[a]$  (or more likely `N[f[a]]`) in Mathematica. How accurate were you?

Mathematica tells us that `N[Sqrt[5]]` is 2.23607, which is pretty close to 2.25. If we want to do better, in a few weeks we will cover a tool called “Newton’s Method” that allows us to refine our answers and get closer approximations. Alternatively, the tool of “Taylor series” gives another improved method of approximations, which uses the both the first and second (and potentially higher) derivatives. We will talk about this in the last week of the course.

## Exercises

Now answer the following questions. In each problem, before you do any computations, think carefully about what you should use for  $f, a, b$ .

1. Estimate  $(2.1)^5$  without doing any calculations. Then use the tangent line to approximate  $(2.1)^5$ . How do these compare? Can you see your original estimate in your tangent line formula?
2. Now approximate  $(2.5)^5$  using  $a = 2$ . Approximate  $3^5$  using  $a = 2$ . Are your approximations getting better or worse? What does this tell you about what counts as “close” to 2?
3. Without calculating, find an upper bound and a lower bound for  $(4.5)^3$ . (Hint:  $4 < 4.5 < 5$ ). Now approximate  $4.5^3$  with a tangent line in two different ways, from two different base points. What happens?
4. Approximate  $\text{Sin} [.05]$  and  $\text{Cos} [.05]$  Note that this is .05 and not .5. Find a formula to approximate  $\sin(x)$  and  $\cos(x)$  when  $x$  is “small”. (This is the revenge of the Small Angle Approximation).
5. Now try  $\text{Sin}[3/4]$  and  $\text{Cos}[3/4]$  (Think about your choices for  $a$  here; you can do much better than 1) .
6. Approximate  $\text{CubeRoot}[28]$  and  $82^{1/4}$ .
7. Now approximate  $28^3$  and  $82^4$  from the same base points. Are these approximations better or worse than your approximations of  $\text{CubeRoot}[28]$  and  $82^{1/4}$  above? Why?
8. Use the same method to find  $\text{CubeRoot}[64.1]$ .
9. Approximate  $(1.01)^{10}$ . Approximate  $(1.01)^\alpha$  where  $\alpha \neq 0$  is some constant (your answer will have an  $\alpha$  in it).  
Now find a formula to approximate  $(1 + \epsilon)^\alpha$  where  $\epsilon$  is a “small” constant and  $\alpha \neq 0$  is a constant. (This rule is called the “binomial approximation” and is often useful in physics).
10. If you take  $a = 0$  and  $f(x) = x^{10}$ , use a tangent line to approximate  $f(2)$ . What happens and why? What if you instead approximate with  $a = 1$ ?