Math 114 Practice Exam 3 Solutions

Problem 1. You may use any methods we have learned in class to solve these problems, but show enough work to justify your answers.

(a) Find $\frac{d^2f}{dx^2}$ if $f(x) = x\cos x$.

Solution:

$$\begin{aligned} f'(x) &= \cos x - x\sin x \\ f''(x) &= -\sin(x) - \sin(x) - x\cos x \\ &= -2\sin(x) - x\cos x \end{aligned}$$

(b) If $g(x) = \sin(3x)$ compute $g'(\pi/12)$

Solution:

$$\begin{aligned} g'(x) &= \cos(3x)\cdot 3. \\ g'(\pi/12) &= \cos(\pi/4)\cdot 3 = 3\sqrt{2}/2 \end{aligned}$$

(c) Find an equation of the line tangent to $y = \frac{x^2-1}{x^2+1}$ at the point $(0,-1)$.

Solution: We have that

$$y' = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

so when $x = 0$ we have $y' = (0-0)/1 = 0$. The equation for a tangent line is $y = m(x-x_0)+y_0$, so the tangent line to this function at $(0,1)$ is $y = 0(x-0)+(-1)$, or $y = -1$.

Problem 2. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)\cdot\tan\left(\frac{\sqrt{x^2+1}}{x+2}\right)\cdot\frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2) - \sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4}\cdot\frac{(3x^2 - \sin(x^2)2x)(\sin(x^3)+1) - \cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

Problem 3. (a) Find a formula for y' in terms of x and y if $x^8 + x^4 + y^4 + y^6 = 1$.

Solution:

$$
\begin{aligned}
8x^7 + 4x^3 + 4y^3\frac{dy}{dx} + 6y^5\frac{dy}{dx} &= 0\\
8x^7 + 4x^3 &= (4y^3 + 6y^5)\frac{dy}{dx}\\
-\frac{4x^7 + 2x^3}{2y^3 + 3y^5} &= \frac{dy}{dx}.
\end{aligned}
$$

(b) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$
\begin{aligned}
4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} &= 0\\
4x^3 - 4xy^2 &= 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}\\
\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} &= \frac{dy}{dx}
\end{aligned}
$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$
\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2\cdot 1 - 4\cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.
$$

Thus the equation of our tangent line is

$$
\begin{aligned}
y - y_0 &= m(x - x_0)\\
y - 1 &= \sqrt{5}(x - \sqrt{5}).
\end{aligned}
$$

(c) If $x^2y = x + y$ find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution: We get

$$
\begin{aligned}
2xy + x^2y' &= 1 + y'\\
(x^2 - 1)y' &= 1 - 2xy\\
y' &= \frac{1 - 2xy}{x^2 - 1}\\
y'' &= \frac{-2(y + xy')(x^2 - 1) - 2x(1 - 2xy)}{(x^2 - 1)^2}\\
&= \frac{-2\left(y + x\frac{1-2xy}{x^2-1}\right)(x^2 - 1) - 2x(1 - 2xy)}{(x^2 - 1)^2}
\end{aligned}
$$

Problem 4. (a) It is a fact that $2^{10} = 1024$. Estimate 2.01^{10} using the derivative of x^{10} at the point 2.

Solution: We set $f(x) = x^{10}$ and see that $f'(x) = 10x^9$. Using our formulas, we then have

$$
\begin{aligned}
f(2.01) &\approx f(2) + (2.01 - 2)f'(2)\\
&= 1024 + .01(10\cdot 2^9) = 1024 + 512/10 = 1024 + 51.2 = 1075.2.
\end{aligned}
$$

(b) Use a linear approximation to estimate $\sqrt{4.01}$.

Solution: Use $f(x) = \sqrt{x}$ and $a = 4$. Then we have

$$
\begin{aligned}
f(4.01) &\approx f'(4)(x - 4) + f(4)\\
&= \frac{1}{4}(.01) + 2 = 2.0025.
\end{aligned}
$$

(c) Suppose we have the differential equation $f'(t) = f(t) - t$, with $f(1) = 2$. Use Euler's method with three steps to approximate $f(4)$.

Solution: We have

$$
\begin{aligned}
f(2) &\approx f'(1)(2-1) + f(1) = 1 + 2 = 3 \\
f(3) &\approx f'(2)(3-2) + f(2) \approx 1(3-2) + 3 = 4 \\
f(4) &\approx f'(3)(4-3) + f(3) \approx 1(4-3) + 4 = 5.
\end{aligned}
$$

Problem 5. (a) The surface area of a cube is given by the formula $A = 6s^2$ where s is the length of a side. If the side lengths are increasing by 2 inches per second, how fast is the surface area increasing when the area is 54 square inches?

Solution: We have the data $A = 6s^2, A = 54, s' = 2$. We take a derivative and see that $A' = 12ss'$, so we need to find s. But when $A = 54$ we have

$$
\begin{aligned}
54 &= 6s^2 \\
9 &= s^2 \\
3 &= s
\end{aligned}
$$

and thus

$$A' = 12ss' = 12 \cdot 3 \cdot 2 = 72$$

so the area is increasing at 72 inches per second.

(b) A car is driving down a road at 150 feet per second (this is about a hundred miles an hour). A camera is placed 200 feet from the road, which will rotate to follow and record the progress of the car. How quickly must the camera rotate when the car is fifty feet away from directly in front of the camera?

Solution: Let the car's position be x, and the angle at which the camera is pointing is θ. Then we have $x = 50, x' = 150$, and we are looking for θ'. We have the equation $\tan\theta = x/200$, and thus

$$
\begin{aligned}
\sec^2\theta \cdot \theta' &= \frac{x'}{200} \\
&= 150/200 = 3/4.
\end{aligned}
$$

But we know that our triangle has sides of length 50 and 200, so the hypotenuse must have length $\sqrt{50^2 + 200^2} = \sqrt{2500 + 40000} = \sqrt{42500} = 10\sqrt{425} = 50\sqrt{17}$. Thus $\sec\theta = \sqrt{17}/4$ and $\sec^2\theta = (\sqrt{17}/4)^2 = 17/16$, and we have

$$
\begin{aligned}
17\theta'/16 &= 3/4 \\
\theta' &= 12/17 \approx .70588.
\end{aligned}
$$

Thus the camera must rotate at 12/17 radians/second.