Math 114 Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to 1} \frac{x^2 - x}{x - 1}$.

Solution: We guess 1.

Let $\epsilon > 0$ and let $\delta \leq \underline{\epsilon}$. Then if $0 < |x - 1| < \delta$ then we have

$$\begin{aligned} \left| \frac{x^2 - x}{x - 1} - 1 \right| &= \left| \frac{(x-1)x}{x - 1} - 1 \right| \\ &= |x - 1| < \delta \leq \epsilon. \end{aligned}$$

(b) Directly from the definition, compute with proof $\lim_{x \to -1} \frac{x + 2}{x}$.

Solution: We guess -1.

Let $\epsilon > 0$ and set $\delta \leq \underline{1/2, \epsilon/4}$. Then if $0 < |x + 1| < \delta$ then

$$\begin{aligned} \left| \frac{x + 2}{x} + 1 \right| &= \left| \frac{2x + 2}{x} \right| \\ &< \frac{2\delta}{|x|}. \end{aligned}$$

To handle the bottom, we observe $|x| = |x + 1 - 1| \geq |1| - |x + 1| > 1 - \delta \geq 1/2$. Thus we have

$$\begin{aligned} \left| \frac{x + 2}{x} + 1 \right| &< \frac{2\delta}{|x|} \\ &< \frac{2\delta}{1/2} = 4\delta \leq \epsilon. \end{aligned}$$

Problem 2.

Let

$$f(x) = \begin{cases} 1 & x < 1 \\ 7 & x > 1 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x \to 4} f(x)$.

Solution: We guess 7.

Let $\epsilon > 0$ and set $\delta = \underline{1}$. Then if $0 < |x - 4| < \delta$, we see that $-1 < x - 4 < 1$ and thus $3 < x < 5$, so $f(x) = 7$, and thus we have

$$|f(x) - 5| = |7 - 7| = 0 < \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x \to 1} f(x)$ does not exist.

Solution: Set $\epsilon = \underline{1}$ and suppose $\delta > 0$. Suppose $\lim_{x\to 1} f(x) = L$. Then set $x_1 = 1+\delta/2, x_2 = 1-\delta/2$, and we have

$$\begin{aligned}\epsilon &> |f(x_1) - L| = |f(1+\delta/2) - L| = |7 - L|\\ \epsilon &> |f(x_2) - L| = |f(1-\delta/2) - L| = |1 - L|\\ 2\epsilon &> |L-1| + |7-L| \geq |L-1+7-L| = |6| = 6\end{aligned}$$

Since $\epsilon = 1$ this gives us $2 > 6$, which is a contradiction.

Problem 3.

Let

$$g(x) = \begin{cases} 4x+1 & x < -2 \\ x+10 & x > -2 \end{cases}$$

(a) Directly from the definition of a limit, compute with proof $\lim_{x\to 0} g(x)$

Solution: We guess 10.

Let $\epsilon > 0$ and let $\delta \leq \underline{2, \epsilon}$. Then if $0 < |x| < \delta$, we see that $-2 < -\delta < x < \delta$ so $x > -2$ so $g(x) = x+10$. Then we have

$$|g(x) - 10| = |x + 10 - 10| = |x| < 2 \leq \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x\to -2} g(x)$ does not exist.

Solution:

Suppose $\lim_{x\to -2} g(x) = L$. Set $\epsilon = \underline{7}$ and let $\delta > 0$. Let $x_1 = -2+\delta/2$ and $x_2 = -2-\delta/2$. Then we have

$$\begin{aligned}\epsilon &> |g(x_1) - L| = |(-2+\delta/2) + 10 - L| = |8 + \delta/2 - L|\\ \epsilon &> |g(x_2) - L| = |4(-2-\delta/2) + 1 - L| = |-7 - 2\delta - L| = |L + 7 + 2\delta|\\ 2\epsilon &> |8+\delta/2 - L| + |L + 7 + 2\delta| \geq |15 + 5\delta/2| = 15 + 5\delta/2 > 15.\end{aligned}$$

Since $\epsilon = 7$ this gives us $14 > 15$, which impossible. So no such limit exists.

Problem 4. (a) Directly from the definition, prove that $\lim_{x\to 1} \dfrac{5}{x-1} = \pm\infty$.

Solution:

Let $N > 0$ and set $\delta = \underline{5/N}$. Then if $0 < |x-1| < \delta$ we have

$$\left|\frac{5}{x-1}\right| = \frac{5}{|x-1|} > \frac{5}{\delta} = \frac{5}{5/N} = N.$$

(b) Directly from the definition, prove that $\lim_{x\to -2} \dfrac{x+4}{(x+2)^2} = +\infty$.

Solution: Let $N > 0$ and set $\delta \leq \underline{1, 1/\sqrt{N}}$. Then if $|x+2| < \delta$ we have that $x+2 > -1$ and thus $x+4 > 1$. Then we have

$$\begin{aligned}\frac{x+4}{(x+2)^2} &> \frac{1}{(x+2)^2} > \frac{1}{\delta^2}\\ &> \frac{1}{(1/\sqrt{N})^2} = N.\end{aligned}$$

Problem 5. Compute the following limits, **showing each step and naming each limit law you use.**

(a)

$$\lim_{x\to 4}(x+1)^2\sqrt{x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 4}(x+1)^2\sqrt{x} &= \left(\lim_{x\to 4}(x+1)^2\right)\left(\lim_{x\to 4}\sqrt{x}\right) && \text{Products}\\
&= \left(\left(\lim_{x\to 4}(x+1)\right)^2\right)\sqrt{\lim_{x\to 4}x} && \text{Exponents}\\
&= \left(\lim_{x\to 4}x+\lim_{x\to 4}1\right)^2\sqrt{\lim_{x\to 4}x} && \text{Additivity}\\
&= \left(4+\lim_{x\to 4}1\right)^2\sqrt{4} && \text{Identity}\\
&= (4+1)^2\sqrt{4}=50 && \text{Constants}
\end{aligned}$$

(b)

$$\lim_{x\to -3}\frac{x^2+2x-3}{2x+6}$$

Solution:

$$\begin{aligned}
\lim_{x\to -3}\frac{x^2+2x-3}{2x+6} &= \lim_{x\to -3}\frac{(x+3)(x-1)}{2(x+3)} && \text{Algebra}\\
&= \lim_{x\to -3}\frac{x-1}{2} && \text{Almost Identical Functions}\\
&= \frac{\lim_{x\to -3}x-1}{\lim_{x\to -3}2} && \text{Quotients}\\
&= \frac{\lim_{x\to -3}x-\lim_{x\to -3}1}{\lim_{x\to -3}2} && \text{Additivity}\\
&= \frac{-3-\lim_{x\to -3}1}{\lim_{x\to -3}2} && \text{Identity}\\
&= \frac{-3-1}{2}=-2 && \text{Constants}
\end{aligned}$$