2 Metric Spaces

2.1 Metrics

Definition 2.1. A metric space is a set E and a function $d : E \times E \to \mathbb{R}$ such that for all $p, q, r \in E$:

1. (Non-negativity) $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$;
2. (Symmetry) $d(p, q) = d(q, p)$;
3. (Triangle inequality) $d(p, r) \leq d(p, q) + d(q, r)$

We say that d is a *metric* on $\mathbb{R}$.

Example 2.2. The most important and fundamental example of a metric space is $\mathbb{R}$. We define our metric to be $d(a, b) = |a - b|$. All three properties of the definition follow directly from the equivalent properties of the absolute value function.

Example 2.3. Let $E = \mathbb{R}^n$ be the set of ordered n-tuples of real numbers, and define

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Then this is a metric space, and we call d the *Euclidean metric*. This is exactly the definition of distance we get doing normal three-dimensional geometry from the Pythagorean Theorem. (Some books, like Rosenlicht, call this space E^n to stand for "Euclidean space").

It's easy to verify the first two properties hold. Non-negativity holds because a sum of squares is always ≥ 0, and is equal to zero if and only if all of the terms are zero. And symmetry holds because $(x_i - y_1)^2 = (y_i - x_i)^2$.

Verifying the triangle inequality is a bit trickier, and depends on the Cauchy-Schwarz inequality, which you may recall from Linear Algebra.

Lemma 2.4 (Cauchy-Schwarz Inequality)**.** *Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be real numbers. Then*

$$|x_1 y_1 + \cdots + x_n y_n| \leq \sqrt{x_1^2 + \cdots + x_n^2} \sqrt{y_1^2 + \cdots + y_n^2}.$$

This may look more familiar in the form

$$\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|.$$

Now we can prove that the Euclidean metric satisfies the triangle inequality:

$$\begin{aligned}(x_1+y_1)^2+\cdots+(x_n+y_n)^2 &= (x_1^2+\cdots+x_n^2)+2(x_1y_1+\cdots+x_ny_n)+(y_1^2+\cdots+y_n^2)\\ &\leq (x_1^2+\cdots+x_n^2)+2\sqrt{x_1^2+\cdots+x_n^2}\sqrt{y_1^2+\cdots+y_n^2}\\ &\qquad +(y_1^2+\cdots+y_n^2)\\ &= \left(\sqrt{x_1^2+\cdots+x_n^2}+\sqrt{y_1^2+\cdots+y_n^2}\right)^2.\end{aligned}$$

Taking square roots gives

$$\sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} \leq \sqrt{x_1^2+\cdots+x_n^2}+\sqrt{y_1^2+\cdots+y_n^2}.$$

Now we can compute

$$\begin{aligned}d(\vec{x},\vec{z}) &= \sqrt{(x_1-z_1)^2+\cdots+(x_n-z_n)^2}\\ &= \sqrt{((x_1-y_1)+(y_1-z_1))^2+\cdots+((x_n-y_n)+(y_n-z_n))^2}\\ &\leq \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}+\sqrt{(y_1-z_1)^2+\cdots+(y_n-z_n)^2}\\ &= d(x,y)+d(y,z).\end{aligned}$$

Example 2.5. Let $E=\mathbb{R}^n$ and let $d(\vec{x}, vecy) = \max\{|x_i-y_i|\}$. We can check this is a metric on E.

1. Let $\vec{x},\vec{y}\in\mathbb{R}^n$. Then $d(\vec{x},\vec{y})\geq 0$ since $|x_i-y_i|\geq 0$ for any real numbers x_i, y_i. $d(\vec{x},\vec{y})=0$ if and only if $|x_i-y_i|=0$ for every i, which happens if and only if $x_i=y_i$ for every i.

2. $d(\vec{x},\vec{y})=\max\{|x_i-y_i|\}$. But $|x_i-y_i|=|y_i-x_i|$, so $\max\{|x_i-y_i|\}=\max\{|y_i-x_i|\}=d(\vec{y},\vec{x})$.

3. Let $\vec{x},\vec{y},\vec{z}\in\mathbb{R}^n$. We know that $d(\vec{x},\vec{y})=\max\{|x_i-y_i|\}$ and $d(\vec{y},\vec{z})=\max\{|y_i-z_i\}$. For each i, we have $|x_i-z_i|\leq|x_i-y_i|+|y_i-z_i|$.

   Then $d(\vec{x},\vec{z})=\max\{|x_i-z_i|\}=|x_j-z_j|$ for some specific j. But we know that

   $$\begin{aligned}|x_j-z_j| &\leq |x_j-y_j|+|y_j-z_j|\\ &\leq \max\{|x_i-y_i|\}+\max\{|y_i-z_i|\}=d(\vec{x},\vec{y})+d(\vec{y},vecz).\end{aligned}$$

Example 2.6. Let E be any set and for every $x, y \in E$, define

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

This is called the *discrete metric* and E is a *discrete metric space.*

1. It's clear that $d(x,y) \geq 0$ since $d(x,y) \in \{0,1\}$. Further, $d(x,y) = 0$ if and only if $x = y$ by definition.

2. $d(x,y) = d(y,x)$ since $x = y$ if and only if $y = x$.

3. Let $x, y, z \in E$. If $x = z$ then $d(x,z) = 0$, and $d(x,y) + d(y,z) \geq 0$, so $d(x,z) \leq d(x,y) + d(y,z)$.

   If $x \neq z$ then $d(x,z) = 1$. But at least one of $x \neq y$ or $y \neq z$, so $d(x,y) + d(y,z) \geq 1$. Thus $d(x,z) \leq d(x,y) + d(y,z)$.

Example 2.7. Let $E = \mathcal{C}([0,1], \mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. For $f, g \in E$, define

$$d(f,g) = \int_0^1 (f(t) - g(t))^2 \, dt.$$

Then this is a metric on E, called the L^2 *metric.*

Proposition 2.8 (Reverse Triangle Inequality)**.** *Let (E,d) be a metric space, and let $p, q, r \in E$. Then $|d(p,r) - d(q,r)| \leq d(p,q)$.*

Proof. Geometrically, this is the claim that the difference of any two sides of a triangle is less than the length of the third side—otherwise the third side wouldn't reach all the way.

Formally we can prove this from the triangle inequality. By the triangle inequality we have

$$\begin{aligned} d(p,r) &\leq d(p,q) + d(q,r) & d(q,r) &\leq d(q,p) + d(p,r) \\ d(p,r) - d(q,r) &\leq d(p,q) & d(q,r) - d(p,r) &\leq d(q,p) \\ d(p,r) - d(r,q) &\leq d(p,q) & -(d(p,r) - d(r,q)) &\leq d(p,q) \end{aligned}$$

and combining these two inequalities gives the desired result. $\square$

2.2 Open and Closed Sets

To understand these metrics, it's helpful to figure out what a circle looks like. A circle captures the idea of equal distances, and everything inside a circle is relatively close to the center. We want to generalize this to any metric.

Definition 2.9. Let E be a metric space, $x_0 \in E$, and $r > 0$ a real number. We define the *open ball* centered at x_0 with radius r as

$$B(x_0, r) = B_r(x_0) = \{x \in E : d(x_0, x) < r\}.$$

We define the *closed ball* centered at x_0 with radius r as

$$\overline{B}(x_0, r) = \overline{B}_r(x_0) = \{x \in E : d(x_0, x) \leq r\}.$$

If our metric space is $\mathbb{R}^3$, then an open or closed ball is just a literal three-dimensional ball; an open ball doesn't contain the spherical boundary, and a closed ball does. In $\mathbb{R}^2$, an open ball is the interior of a circle, and the closed ball is the circle including its boundary. In some other metrics, the balls are more unusual.

In $\mathbb{R}$ these concepts are even more familiar. The open ball $B_r(x_0)$ is the open interval $(x_0 - r, x_0 + r)$, and the closed ball $\overline{B}_r(x_0)$ is the closed interval $[x_0 - r, x_0 + r]$.

Example 2.10. If E is a metric space under the discrete metric, then $B_r(x_0) = \{x_0\}$ for $r \leq 1$ and $B_r(x_0) = E$ for $r > 1$.

$\overline{B}_r(x_0) = \{x_0\}$ for $r < 1$ and $\overline{B}_r(x_0) = E$ for $r \geq 1$.

Example 2.11. If $E = \mathbb{R}^2$ under the sup metric, then $B_r(x_0)$ is a square centered at x_0 with side length $2r$.

Example 2.12. If $E = \mathcal{C}([0, 1], \mathbb{R})$ is the space of continuous functions from $[0, 1]$ to $\mathbb{R}$ under the L^2 metric, then the open ball centered at the 0 function of radius 1 is the set of all functions f such that $\int_0^1 f(x)^2 \, dx < 1$.

We see a clear distinction between open balls and closed balls: closed balls contain their boundaries, and open balls do not. (And we can imagine a set that contains *some* of its boundary and is thus neither open nor closed). We'd like to extend this distinction to all sets, not just the ones that are shaped like sphers.

Definition 2.13. Let E be a metric space and $U \subset E$. We say that U is an *open set* if for each $x \in U$, there is some $r > 0$ such that $B_r(x) \subset U$.

Example 2.14. Consider the space $E = \mathbb{R}^2$ with the sup metric. Let $U = \{(x, y) : x > 0\}$. Let's prove that U is open.

Let $(x, y) \in U$. We claim that $B_x(x, y) \subset U$. To prove this, let $(x_1, y_1) \in B_x(x, y)$; we want to show that $(x_1, y_1) \in U$, that is, that $x_1 > 0$.

Since $(x_1, y_1) \in B_x(x, y)$, we know that $d((x_1, y_1), (x, y)) < x$. Thus $\max\{|x_1 - x|, |y_1 - y|\} < x$, and in particular $|x_1 - x| < x$.

From here we can make one of two arguments. One is that $-x < x_1 - x < x$, so $0 < x_1 < 2x$. The other is that by the reverse triangle inequality, we have $x > |x - x_1| \geq x - x_1$, and thus $x_1 > 0$. Either way, we see that $x_1 > 0$, so $(x_1, y_1) \in U$.

This proves that $B_x(x, y) \subset U$. Since $(x, y) \in U$ was arbitrary, we conclude that U is open.

Proposition 2.15. *Let E be a metric space. Then:*

1. *$\varnothing$ and E are open sets.*
2. *An arbitrary union of open sets is open.*
3. *A* finite *intersection of open sets is open.*

Proof.

1. For every $x \in \varnothing$, anything is true, since there are no $x \in \varnothing$. Thus for every $x \in \varnothing$ there is an $r > 0$ such that $B_r(x) \subset \varnothing$. Thus $\varnothing$ is open.

   For every $x \in E$ and every $r > 0$, $B_r(x) \subset E$ by definition. Thus E is open.

2. Let $\{U_\alpha\}$ be a collection of open sets, and let $x \in \bigcup U_\alpha$. Then $x \in U_\beta$ for some β, and since U_β is open, there is a $r > 0$ such that $B_r(x) \subset U_\beta$. But $U_\beta \subset \bigcup U_\alpha$, so $b_r(x) \subset \bigcup U_\alpha$.

3. Let $U_1, \dots, U_n$ be open sets and let $x \in \bigcap U_i$. Then $x \in U_i$ for each i, and thus there is a $r_i > 0$ such that $B_{r_i}(x) \subset U_i$.

   Let $r = \min\{r_1, \dots, r_n\}$. Then $B_r(x) \subset B_{r_i}(x)$ for every i, so $B_r(x) \subset U_i$ for each i. Thus $B_r(x) \subset \bigcap U_i$. We conclude that $\bigcap U_i$ is open.

□

Notice that while openness is preserved for any union, it is only preserved for finite intersections. (We needed finiteness when we took the minimum of the set—an infinite collection might have no minimum, and the greatest lower bound might be zero, which doesn't work). In your homework you will look at an example of an infinite intersection of open sets that isn't open.

Proposition 2.16. *Let E be a metric space, let $x \in E$, and $r > 0$. Then $B_r(x)$ is an open set.*

Proof. We need to prove that any point in $B_r(x)$ has an open ball around it that is inside $B_r(x)$. We're essentially going to use the triangle inequality to do this: if $y \in B_r(x)$ and you're close enough to y, you also have to be reasonably close to x.

So let $y \in B_r(x)$. Then set $s = d(x, y)$; we know that $s < r$ by definition of an open ball. We claim that $B_{r-s}(y) \subset B_r(x)$.

Let $z \in B_{r-s}(y)$. Then $d(y, z) < r - s$. But $d(x, y) = s$, so by the triangle inequality we have $d(x, z) \leq d(x, y) + d(y, z) < s + r - s = r$. Thus $z \in B_r(x)$. □

Definition 2.17. If E is a metric space, we say that $V \subset E$ is *closed* if the complement of V in E, denoted V^C, is open.

Exercise 2.18. *If E is a metric space, $x \in E$, and $r > 0$, then $\overline{B}_r(x)$ is a closed set.*

Proposition 2.19.

1. *$\varnothing$ and E are closed sets.*
2. *Any intersection of closed sets is closed.*
3. *Any* finite *union of closed sets is closed.*

Proof. This is basically a corollary to proposition 2.33.

1. E is the complement of $\varnothing$ and $\varnothing$ is the complement of E.
2. Let $\{V_\alpha\}$ be a collection of closed sets. For each V_α, the complement U_α is open, so $\bigcup U_\alpha$ is open. $\bigcap V_\alpha$ is the complement of $\bigcup U_\alpha$.
3. Let $V_1, \ldots, V_n$ be closed. For each V_i, the complement U_i is open, so $U_1 \cap \cdots \cap U_n$ is open. $V_1 \cup \cdots \cup V_n$ is the complement of $U_1 \cap \cdots \cap U_n$.

□

Remark 2.20. There's one important subtlety to be aware of: whether a set is open depends not only on the set, but on the metric space it's a part of. This is really clear when we change the metric: the set $\{0\} \subset \mathbb{R}$ is open in the discrete metric, but not in the Euclidean metric.

But especially weird things happen on boundaries. Let $E = [0, 1]$ be a metric space under the metric $d(x, y) = |x - y|$ inherited from the real line. Then it's pretty clear that $B_{1/2}(0)$ is open. But $B_{1/2}(0) = [0, 1/2)$, so in this metric space, $[0, 1/2)$ is open. In contrast, in $\mathbb{R}$, the set $[0, 1/2)$ is neither open nor closed.

Definition 2.21. Let (E, d) be a metric space. We say a subset $U \subset E$ is *bounded* if there is some open or closed ball $B_r(x)$ or $\overline{B}_r(x)$ such that $U \subset B_r(x)$.

Example 2.22. A subset of $\mathbb{R}$ is bounded if and only if it is bounded above and below. If $S \subset \mathbb{R}$ with a as a lower bound and b as an upper bound, then $S \subset [a, b]$. But $[a, b]$ is a closed ball centered at $(a + b)/2$.

Example 2.23.

Conceptually, an open set is a set that doesn't contain any of its boundary; a closed set is a set that contains its entire boundary. We can't use this as a definition since we haven't defined what a "boundary" is yet. But we *can* somewhat justify this for $\mathbb{R}$ where the boundary of a set is just a maximum or minimum element.

Proposition 2.24. *A non-empty closed subset of* $\mathbb{R}$ *that is bounded above contains a greatest element.*

A non-empty closed subset of $\mathbb{R}$ *that is bounded below contains a least element.*

Proof. We'll prove the first claim; the proof of the second is identical.

Let S be a non-empty closed subset of $\mathbb{R}$ that is bounded above. Then S has a least upper bound, so let $a = \sup(S)$. If $a \in S$ then it is the greatest element, so instead suppose $a \in S^C$ the complement of S.

S^C is open by definition, so there is some $r > 0$ such that $B_r(a) \in S^C$. That is, $(a - r, a + r) \in S^C$, so no element of S can be in $(a - r, a + r)$. This implies that $a - r$ is an upper bound for S, but $a - r < a = \sup(S)$, which is a contradiction. So $a \in S$ is a largest element. □

2.3 Convergent Sequences

In this section we want to define the fundamental idea of calculus, which is the idea of convergence to a limit.

Recall a sequence is just an ordered list of things. (If we're being extremely formal, we say a sequence of elements of S is a function from the natural numbers to S; the nth element of the sequence is $f(n)$).

Intuitively, we want to use the word "limit" to describe a point that a sequence gets really close to. People sometimes say "gets closer and closer", but this is wrong for two reasons. First, because a sequence can oscillate around a point and not get strictly closer. Second, because it's entirely possible to get continually closer to a value but not get anywhere near it; we'd never say that the limit of the sequence $x_n = 1/n$ is -3, even though the values of the sequence do get closer and closer to -3.

We can close these loopholes by giving a formal definition.

Definition 2.25. Let (E, d) be a metric space and $x_1, x_2, \ldots$ be a sequence of elements of E. We say that the sequence (x_n) *converges* to a limit $L \in E$, or that $lim_{n\to\infty} x_n = L$, if for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that, whenever $n > N$, then $d(x_n mL) < \epsilon$. Symbolically:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \ni (n > N \Rightarrow d(x_n, L) < \epsilon)$$

Notice that N depends on ϵ, and n depends on N. We don't get any control over the value of ϵ, but we can pick any value of N depending on ϵ to make this definition work.

Convergence depends on the specific metric space. First, because it often depends on the specific metric used. (Very few sequences converge in the discrete metric). Second, because it depends on the points in the space; the sequence $3, 3.1, 3.14, \ldots$ has no limit in the rationals and thus doesn't converge.

Example 2.26. Let's prove that $\lim_{n\to\infty} \frac{3}{n} = 0$.

Let $\epsilon > 0$, and set $N = \frac{3}{\epsilon}$. Then if $n > N$, we have

$$d(3/n, 0) = \left|\frac{3}{n}\right| = \frac{3}{n} < \frac{3}{N} = \frac{3}{3/\epsilon} = \epsilon.$$

Thus by definition, $\lim_{n\to\infty} \frac{3}{n} = 0$.

From this definition, we can prove lots of nice properties for limits. The first is that they're unique—the same sequence can't have more than one limit.

Proposition 2.27. *Let (E, d) be a metric space. Then a sequence (x_n) in E has at most one limit.*

Proof. This proof uses a common trick which I think of as the $\epsilon/2$ or ϵ/n trick. Heuristically, we want to show that if x_n gets arbitrarily close to L and also to K, then L and K have to be arbitrarily close together, and thus identical. We want to show that $d(L, K) < \epsilon$; to do this, we show that two other distances are less than $\epsilon/2$.

Suppose $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} x_n = K$. Let $\epsilon > 0$. Then by definition of limit, there is a $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $d(x_n, L) < \epsilon/2$. And there is a $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $d(x_n, K) < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then if $n < N$, we know that $d(x_n, L) < epsilon/2$ and $d(x_n, K) < \epsilon/2$. Adding these inequalities tells us that

$$d(x_n, L) + d(x_n, K) < \epsilon$$

and the triangle inequality tells us that

$$d(L, K) \leq d(x_n, L) + d(x_n, K).$$

Thus $d(L, K) < \epsilon$.

But ϵ was an arbitrary positive real number, so $d(L, K) = 0$ and thus $L = K$. ☐

An important property of limits of sequences is that you can distort them in a lot of ways without changing the limit. One example is that for any finite $k \in \mathbb{N}$, you can change the first k terms of the sequence without changing the limit; you just have to make sure that you take $N > k$. But we can also make infinite changes if we're careful.

Definition 2.28. If $x_1, x_2, \dots$ is a sequence and $n_1, n_2, \dots$ is a strictly increasing sequence of integers, we say the sequence of elements $x_{n_1}, x_{n_2}, \dots$ is a *subsequence* of (x_n).

Proposition 2.29. *If a sequence (x_n) converges to a limit L, then any subsequence of (x_n) also converges to L.*

Proof. Suppose $\lim_{n\to\infty} x_n = L$. Let $\epsilon > 0$. Then there is a $N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, L) < \epsilon$.

Since $n_k \geq k$ for any $k \in \mathbb{N}$, if $k > N$ then we know that $n_k > N$, and so $d(x_{n_k}, L) < \epsilon$.

Thus by definition of limit, $\lim_{k\to\infty} x_{n_k} = L$. ☐

Exercise 2.30. *Let (E, d) be a metric space, and suppose $x_1, x_2, \dots$ is a sequence that converges to x in E (that is, $\lim_{n\to\infty} x_n = x$). Then the sequence $x_1, x, x_2, x, x_3, \dots$ converges to x.*

Remark 2.31. We can reframe the idea of limits in terms of open sets. It's clear enough that $\lim_{n\to\infty} x_n = L$ if for every open ball $B_\epsilon(L)$ centered at L, there is some $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(L)$ for all $n > N$—this is just the original definition rephrased.

It's not too much harder to convince yourself that $\lim_{n\to\infty} x_n = L$ if and only if, for every open set U with $L \in U$, there is some $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

(In the field of topology, we have a definition of "open set" but not of "metric" or distance. In topology we can take this to be the definition of limit).

We've seen how to prove a specific sequence converges. We can also prove that one doesn't:

Example 2.32. Let $(x_n) = 1, 0, 1, 0, \ldots$ be a sequence of real numbers. Then (x_n) has no limit.

Suppose $\lim_{n\to\infty} x_n = x$. Then for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ so that if $n > N$, then $d(x_n, L) < \epsilon$.

But for some even $n_1 > N$, we have $x_{n_1} = 0$, so $d(0, L) < \epsilon$. And for some odd $n_2 > N$, we have $x_{n_2} = 1$, so $d(1, L) < \epsilon$. By the triangle inequality, we have

$$d(0,1) \leq d(1,L) + d(0,L) < 2\epsilon$$

and thus $1 < 2\epsilon$ for every $\epsilon > 0$. Since this is false, we have a contradiction, so no limit can exist.

2.4 Closures, Boundaries and Interiors

We said earlier that we want to think of a closed set as "containing its boundary," and an open set as not doing that. Sequences and their convergence lets us define this more carefully.

We first need to see how sequences behave in open and closed sets.

Proposition 2.33. *Let (E, d) be a metric space, and let $V \subset E$. Then V is closed if and only if every convergent sequence of points in V converges to a point in V.*

Proof. First, suppose V is closed. Let (x_n) be a sequence contained in V that converges to x, and suppose $x \notin V$. Then x is in the complement of V, which is open. Thus there is some ball $B_\epsilon(x) \subset V^C$.

By definition of convergence, there is some $N \in \mathbb{N}$ such that when $n > N$, we know that $d(x_n, x) < \epsilon$. Thus $x_n \in B_\epsilon(x) \subset V^C$, which is a contradiction since $x_n \in V$.

Now suppose V is not closed. This means that V^C is not open, and so there is some point $x \in V^C$ such that $B_\epsilon(x)$ is not a subset of V^C for any $\epsilon > 0$.

For each $n \in \mathbb{N}$, let $x_n \in B_{1/n}(x)$ such that $x \in V$. This is possible since $B_{1/n}(x) \not\subset V^C$. We claim that $\lim_{n\to\infty} x_n = x$.

Let $\epsilon > 0$. Then there is some $N \in \mathbb{N}$ with $1/N < \epsilon$. if $n < N$, then $x_n \in B_{1/n}(x)$ so $d(x_n, x) < 1/n < 1/N < \epsilon$. Thus $\lim_{n\to\infty} x_n = x$.

So if V is not closed, we can construct a sequence of points $x_n \in V$ that converges to a limit not in V. If no such sequence exists—if every sequence of points in V that converges has a limit in V—then V must be closed. This completes the proof. $\square$

We now want to take sets and find ways to turn them into the closest closed and open sets we can.

Definition 2.34. Let (E, d) be a metric space, and $U \subset E$. We define the *closure* of U to be $\overline{U}$ the intersection of every closed set containing U.

It's clear that $\overline{U}$ contains U, since it's the intersection of sets which all contain U. It's also clear that $\overline{U}$ is closed, since any intersection of closed sets is closed. Less obvious is the folloiwng result:

Proposition 2.35. *Let (E, d) be a metric space and $U \subset E$. Then $\overline{U}$ is the set of the limits of all sequences in U that converge in E.*

Proof. First we show that the set of all limits of sequences in U is contained in the closure of U. Suppose (x_n) is a sequence in U, and $\lim_{n\to\infty} x_n = x$. Then for any closed set V containing U, we have $x_n \in V$ for all n, and by proposition 2.33 we have $x \in V$. Thus x is in every closed set containing U, and so in their intersection.

Now suppose $x \in \overline{U}$; we wish to show that x is the limit of some sequence of points in U. Let $n \in \mathbb{N}$. If $B_{1/n}(x) \cap U = \varnothing$, then $U \subset B_{1/n}(x)^C$; and since $B_{1/n}(x)^C$ is closed, this implies that $\overline{U} \subset B_{1/n}(x)^C$. But $x \in \overline{U}$, so this is impossible.

Thus $B_{1/n}(x) \cap U$ is non-empty. Now as in proposition 2.33 we can choose $x_n \in B_{1/n}(x) \cap U$, and $\lim_{n\to\infty} x_n = x$. □

The closure of U is the smallest closed set containing U; you can think of it as taking U and then adding the "boundary" (which we still haven't defined!).

Exercise 2.36. *Let (E, d) be a metric space, and let V be a closed subset of E. Prove that $\overline{V} = V$.*

Example 2.37. Consider $\mathbb{R}^2$ under the Euclidean metric. Then the closure of $B_1(0, 0)$ is the closed ball $\overline{B}_1(0, 0)$.

Clearly the closure is a subset of $\overline{B}_1(0, 0)$, since $\overline{B}_1(0, 0)$ is a closed set containing $B_1(0, 0)$, and the closure is the intersection of all of these.

Now suppose $x = (x_1, x_2) \in \overline{B}_1(0, 0)$. We want to prove that x is the limit of some sequence (x_n) that is contained in $\overline{B}_1(0, 1)$.

For each n, define $x_n = ((n-1)x/n, (n-1)y/n)$. Then

$$d(x_n, \vec{0}) = \sqrt{\frac{n-1}{n}}\sqrt{x_1^2 + x_2^2} \leq \sqrt{\frac{n-1}{n}} < 1$$

so $x_n \in B_1(0,0)$.

We claim that $\lim_{n\to\infty} x_n = x$. Let $\epsilon > 0$, and let $N > 1/\epsilon$. Then if $n > N$ we compute

$$\begin{aligned} d(x_n, x) &= \sqrt{(x_1(n-1)/n - x_1)^2 + (x_2(n-1)/n - x_2)^2} \\ &= \sqrt{(x_1/n)^2 + (x_2/n)^2} = \frac{1}{n}\sqrt{x_1^2 + x_2^2} = \frac{1}{n} d(x, \vec{0}) \\ &\leq \frac{1}{n} < \frac{1}{N} < \epsilon. \end{aligned}$$

We can also do the equivalent for open sets: take a set and remove the boundary.

Definition 2.38. Let (E,d) be a metric space, and $U \subset E$. We define the *interior* of U to be $\mathring{U}$ the union of every open subset of U.

As before, it's clear that this is an open set, since it is a union of open sets, and it's a subset of U, since it's a union of subsets.

Exercise 2.39. *If (E,d) is a metric space and $U \subset E$, prove the interior of U is the set of all points $x \in U$ such that some open ball containing x is also a subset of U.*

Lemma 2.40. *If U is a set in a metric space (E,d), then $\left(\mathring{U}\right)^C = \overline{U^C}$.*

Proof.

$$\left(\mathring{U}\right)^C = \left(\bigcup_{S \subset U \text{ open}} S\right)^C = \bigcap_{S \subset U \text{ open}} S^C = \bigcap_{V \supset U \text{ closed}} V = \overline{U}.$$

□

Definition 2.41. The *boundary* of a set U is $\partial U = \overline{U} \cap \overline{U^C}$.

Equivalently we can write that $\partial U = \overline{U} \setminus \mathring{U}$, since

$$\overline{U} \setminus \mathring{U} = \overline{U} \cap \left(\mathring{U}\right)^C = \overline{U} \cap \overline{U^C}.$$

Proposition 2.42. *Let (E,d) be a metric space and $S \subset E$. Then $x \in \partial S$ if and only if every ball centered at x contains points in S and also points in S^C.*

Proof. Suppose $x \in \partial S$. Then $x \in \overline{S}$, so x is the limit of some sequence x_n of points in S such that $\lim_{n\to\infty} x_n = x$. Similarly, $x \in \overline{S^C}$, so x is the limit of some sequence of points y_n in S^C such that $\lim_{n\to\infty} y_n = x$.

Then for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that $n > N$ implies $x_n \in B_\epsilon(x)$. And there is some $M \in \mathbb{N}$ so that $m > M$ implies $y_m \in B_\epsilon(x)$. Thus any open ball centered at x contains (infinitely many) points in S and also (infinitely many) points in S^C.

Conversely, suppose any open ball centered at x contains points in S and also in S^C. Then for every $n \in \mathbb{N}$ there is a $x_n \in B_{1/n}(x) \cap S$ and a $y_n \in B_{1/n}(x) \cap S^C$. Then $\lim_{n\to\infty} x_n = x$, so $x \in \overline{S}$. But $\lim_{n\to\infty} y_n = x$ so $x \in \overline{S^C}$. □